ECE 342
Electronic Circuits

Lecture 20
Transfer Functions - 1

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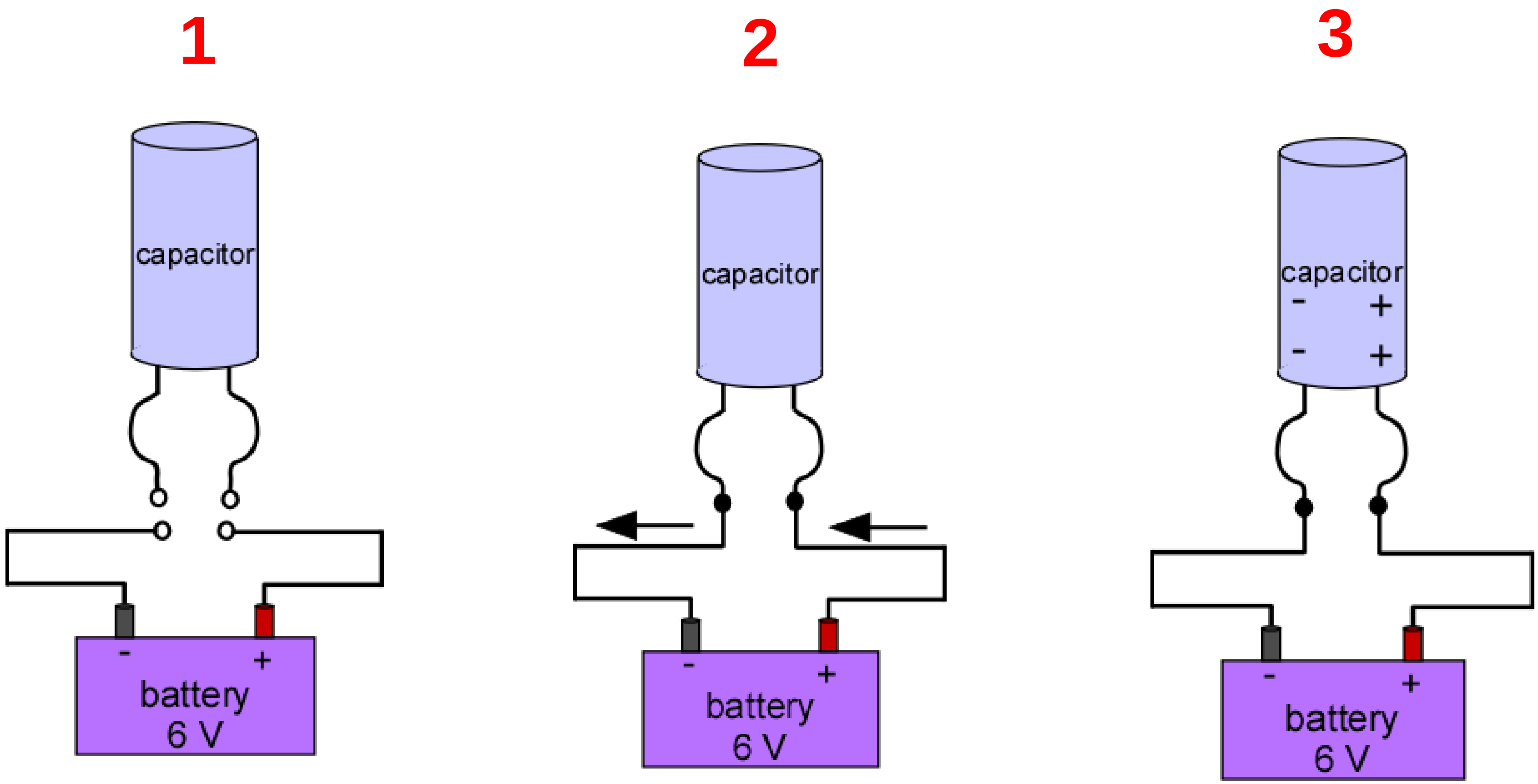
What is Capacitance?

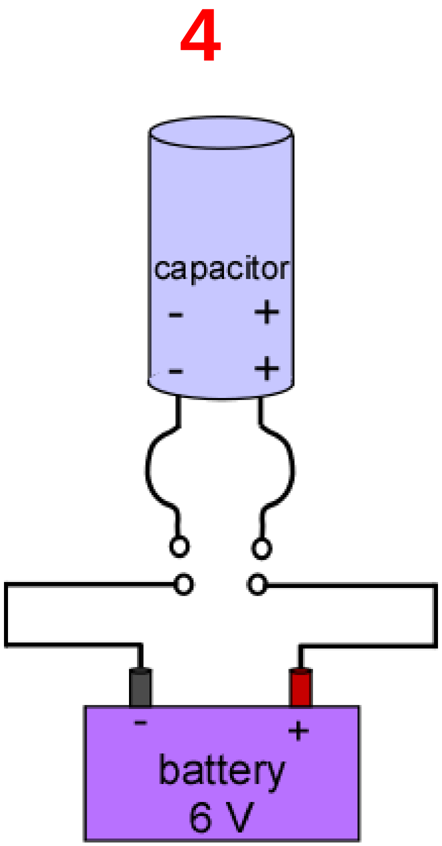


Voltage=0
No Charge
No Current

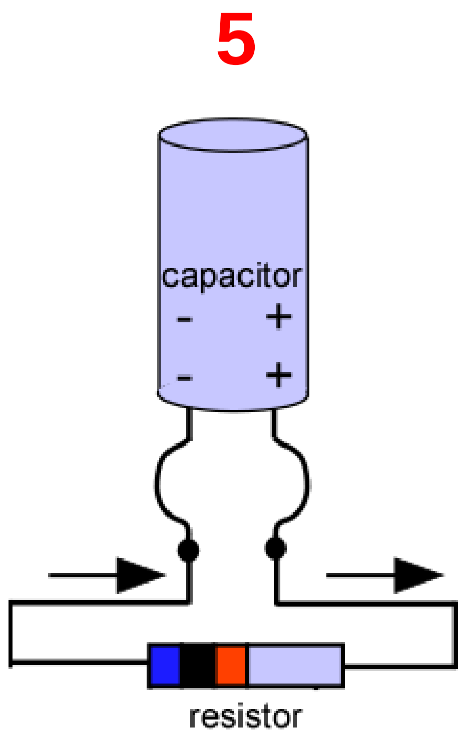
Voltage build up
Charge build up
Transient Current

Voltage = 6V
Charge=Q
No Current

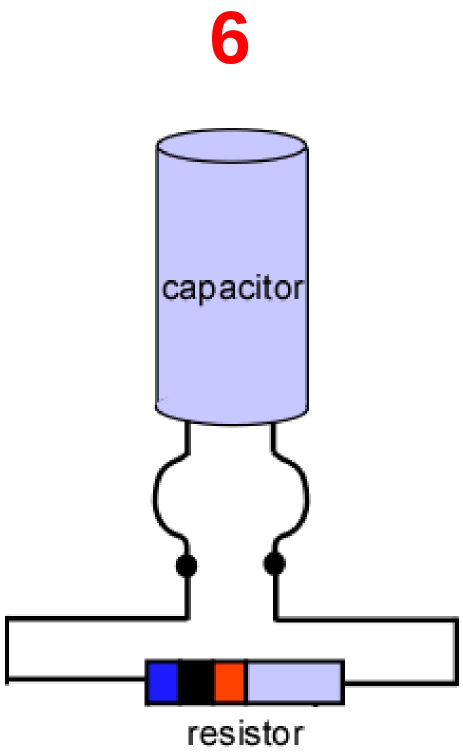
What is Capacitance?



Voltage=6V
Charge=Q
No Current



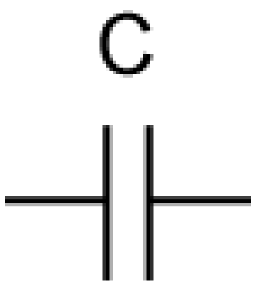
Voltage decaying
Charge decaying
Transient Current



Voltage=0
No Charge
No Current

Capacitance

Relation: $Q = Cv$

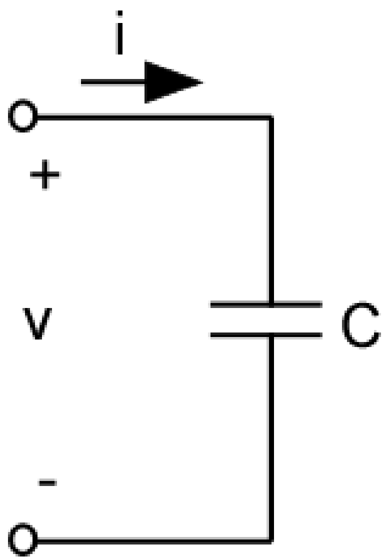


Q: charge stored by capacitor

C: capacitance

v: voltage across capacitor

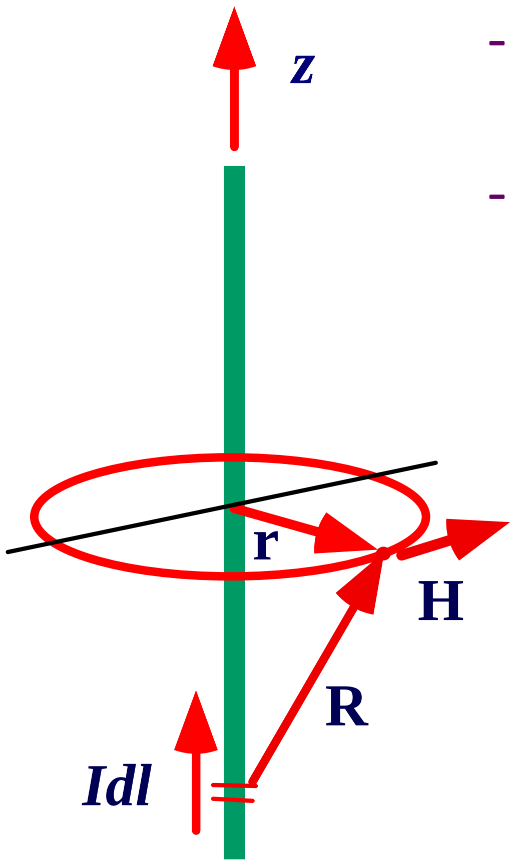
i: current into capacitor



$$i(t) = C\frac{dv}{dt} = \frac{dQ}{dt}$$

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

What is Inductance?

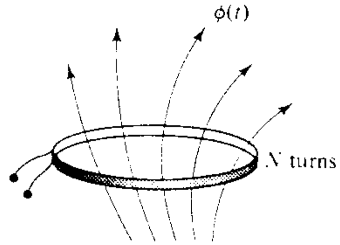


- *Current in wire produces magnetic field*

- *Flux is magnetic field integrated over area*

$$\text{Inductance} = \frac{\textit{Total Flux Linked}}{\textit{Current}}$$

Inductance



$$L = N\frac{d\Psi}{di}$$

$$L = \frac{\Psi}{I}$$

$$L = \int_{v} \frac{\vec{B}\cdot\vec{H}dv}{I^{2}}$$

Inductance

Relation: $\Psi = Li$

Ψ: flux stored by inductor

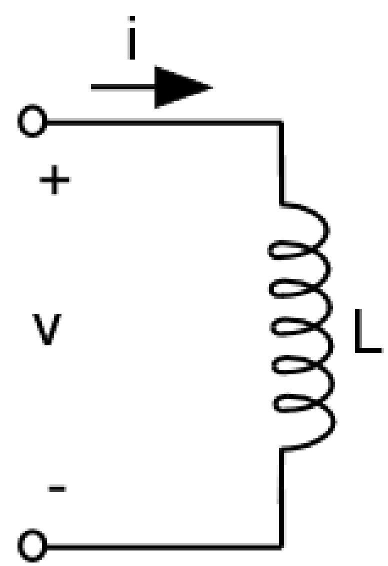
L: inductance

i: current through inductor

v: voltage across inductor

$$v(t) = L\frac{di}{dt} = \frac{d\Psi}{dt}$$

$$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$$

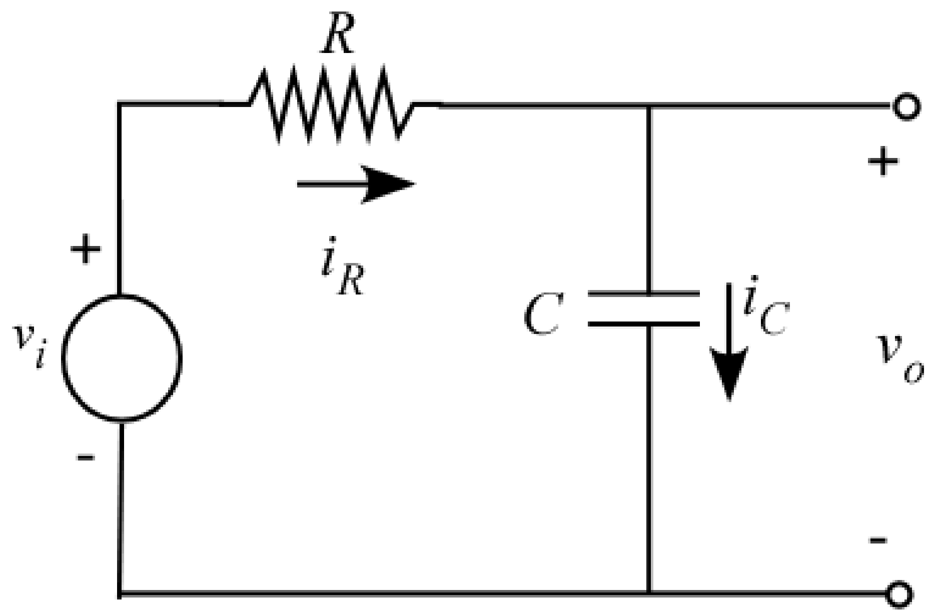


Low-Pass Circuit

$$i_R(t) = C\frac{dv_o}{dt} = i_C(t)$$

$$v_i(t) = Ri_R(t) + v_o(t)$$

$$v_i(t) = RC\frac{dv_o}{dt} + v_o(t)$$



Need to solve for $v_o(t)$

Low-Pass Circuit

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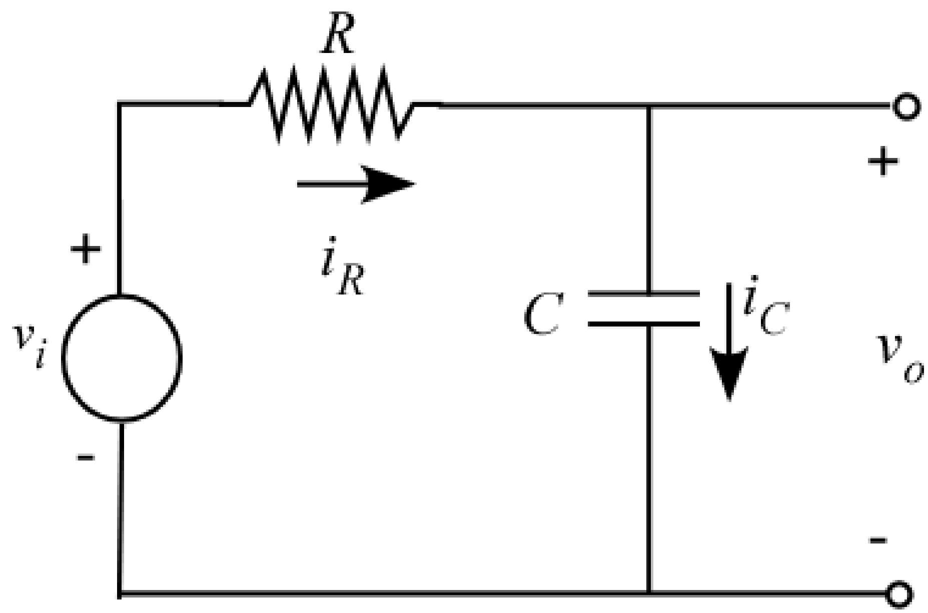
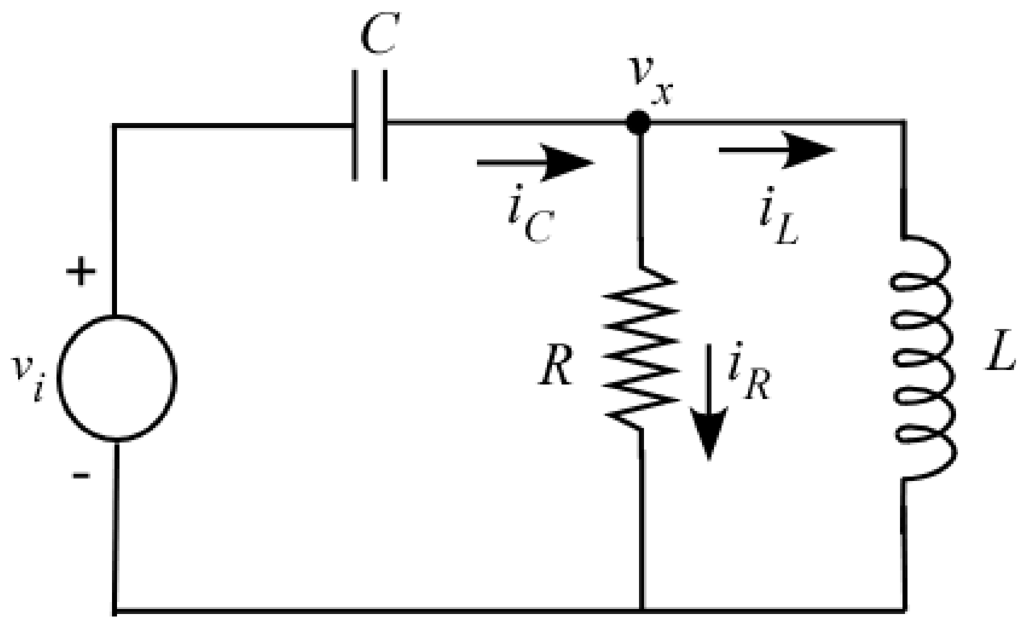


Need to solve for *$v_o(t)$*

Reactive Circuit



$$i_C(t) = i_R(t) + i_L(t)$$

Need to solve for $v_x(t)$

$$v_x = L\frac{di_L}{dt} \qquad i_C = C\frac{d\left(v_i - v_x\right)}{dt}$$

Laplace Transforms

The Laplace transform *F(s)* of a function *f(t)* is defined as:

$$L\{f(t)\} = \int_{0}^{\infty} f(t) e^{-st} dt = F(s)$$

To a mathematician, this is very meaningful; however circuit engineers seldom use that integral

The Laplace transform provides a conversion from the time domain into a new domain, the s domain

Laplace Transforms

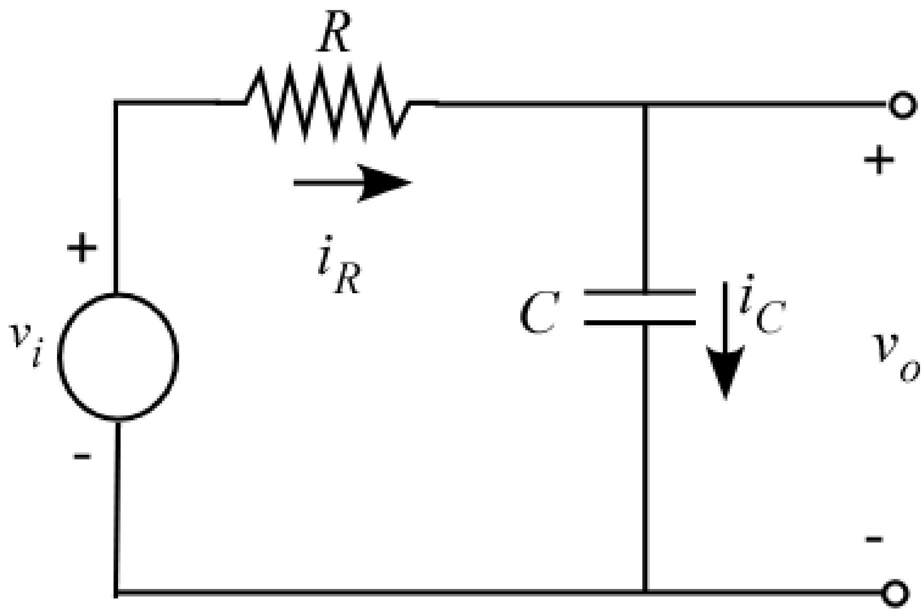
The Laplace transform of the derivative of a function *f(t)* is given by

$$L\left\{\frac{df(t)}{dt}\right\}=\int_0^{\infty}\frac{df(t)}{dt}e^{-st}dt=sF(s)$$

Differentiation in the time domain becomes a multiplication in the s domain. <u>That is why Laplace transforms are useful</u> to circuit engineers

$$s=j\omega$$

Low-Pass Circuit



Time domain | s domain

$$v_o(t) \leftrightarrow V_o(s)$$

$$\frac{dv_o(t)}{dt} \leftrightarrow sV_o(s)$$

Time domain

s domain

$$v_i(t) = RC\frac{dv_o}{dt} + v_o(t) \longleftrightarrow V_i(s) = sRCV_o(s) + V_o(s)$$

Low-Pass Circuit - Solution

$$V_o(s) = \frac{V_i(s)}{1+sRC} \Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{1}{1+sRC}$$

Pole exists at $s = -1/RC$

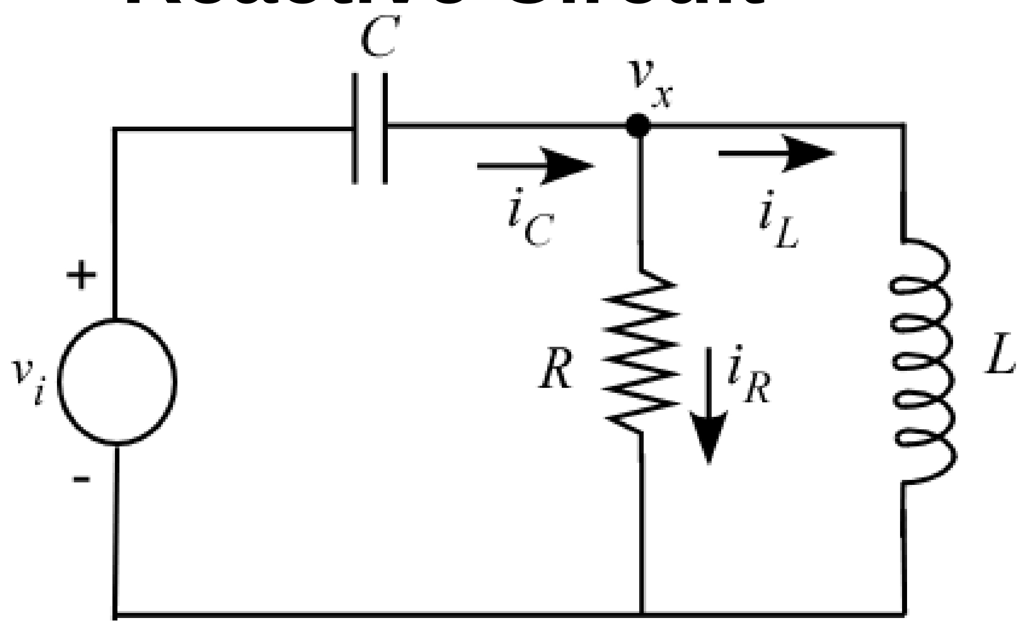
Assume that $V_i(s)=V_d/s$

$$V_o(s) = \frac{V_d}{s\left(1+sRC\right)} = \frac{A}{s} + \frac{B}{\left(s+1/RC\right)}$$

Inversion

$$v_o(t) = Au(t) + Be^{-t/RC}$$

Reactive Circuit



$$v_x = L\frac{di_L}{dt}$$

$$V_x(s) = sLI_L$$

$$i_C = C\frac{d\left(v_i - v_x\right)}{dt}$$

$$I_C = sC(V_i - V_x)$$

Reactive Circuit - Solution

$$sC(V_i - V_x) = \frac{V_x}{R} + \frac{V_x}{sL}$$

$$sCV_i = V_x\left(sC + \frac{1}{R} + \frac{1}{sL}\right)$$

$$V_x = \frac{sCV_i}{\left(sC + \frac{1}{R} + \frac{1}{sL}\right)} = \frac{V_i}{\left(1 + \frac{1}{sRC} + \frac{1}{s^2LC}\right)}$$

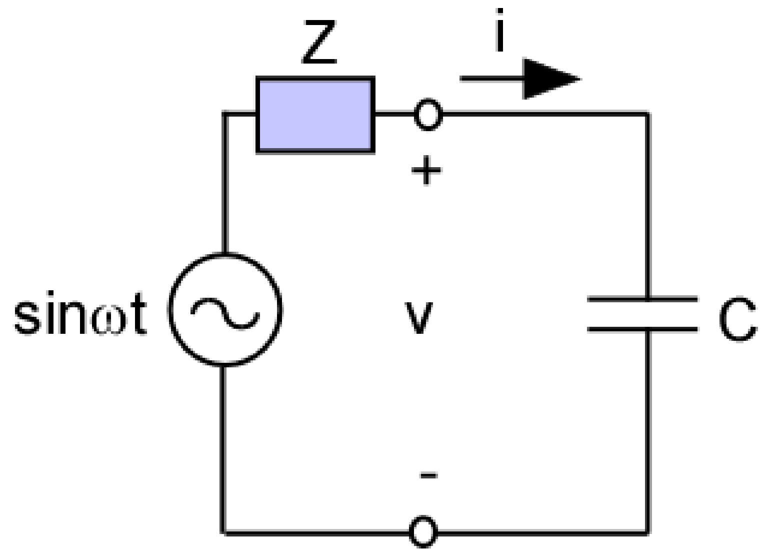
Exercise: Use inversion to solve for $v_x(t)$

Table of Laplace Transforms

Function	Transform
$u(t-\alpha)$	$\frac{1}{s}e^{-\alpha s}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$

Capacitance – s Domain

Assume time-harmonic behavior of voltage and current variables $j\omega = s$



$$i(t), v(t) \sim e^{j\omega t}$$

$$i(t) \leftrightarrow I(s)$$

$$v(t) \leftrightarrow V(s)$$

$$\frac{d}{dt} \leftrightarrow s$$

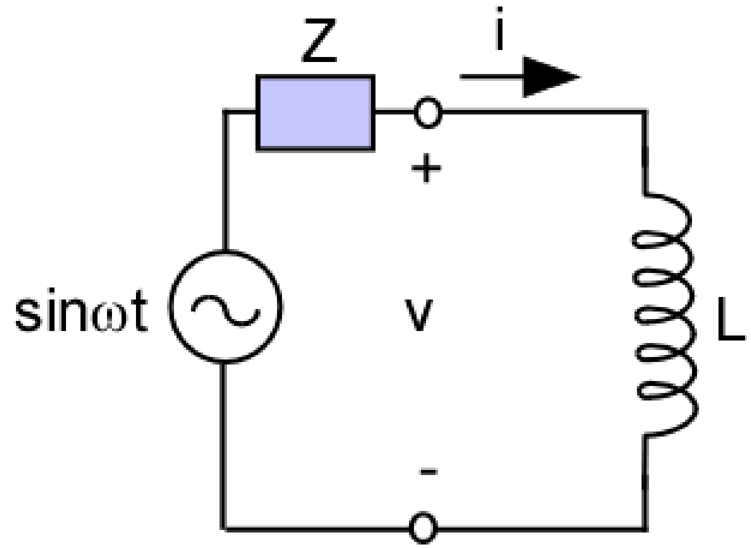
$$I(s) = sCV(s)$$

The capacitor is a reactive element

$$\text{Reactance}: X_C = \frac{1}{sC}$$

Inductance – s Domain

Assume time-harmonic behavior of voltage and current variables $j\omega = s$



$$i(t), v(t) \sim e^{j\omega t}$$

$$i(t) \leftrightarrow I(s)$$

$$v(t) \leftrightarrow V(s)$$

$$\frac{d}{dt} \leftrightarrow s$$

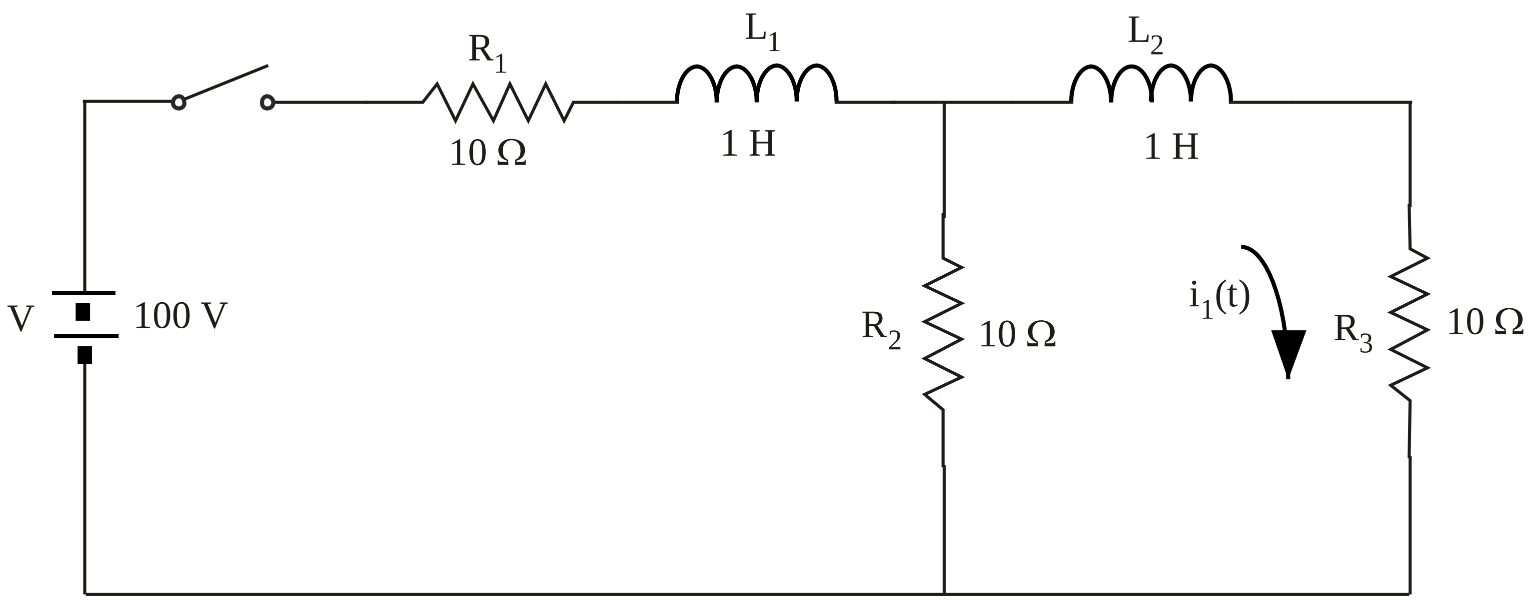
$$V(s) = sLI(s)$$

The inductor is a reactive element

$$\text{Reactance}: X_L = sL$$

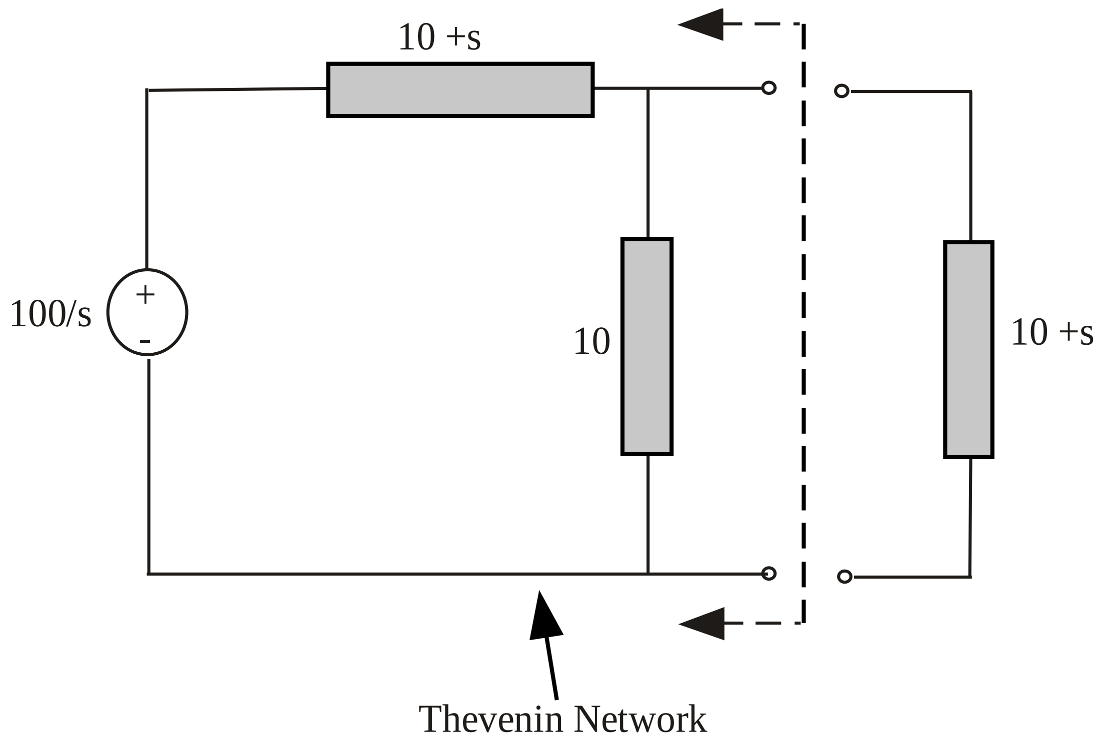
Transfer Function

Need to find the current $i_1(t)$ in resistor R_3



Transfer Function

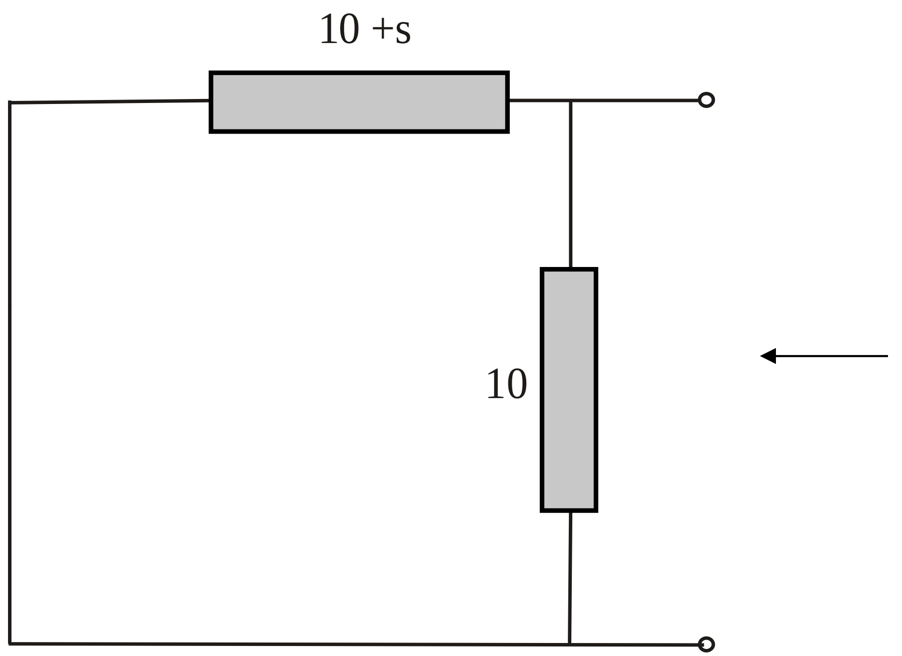
Thevenin voltage calculation



$$V_{th} = \frac{10(100/s)}{10+s+10} = \frac{1,000}{s(s+20)}$$

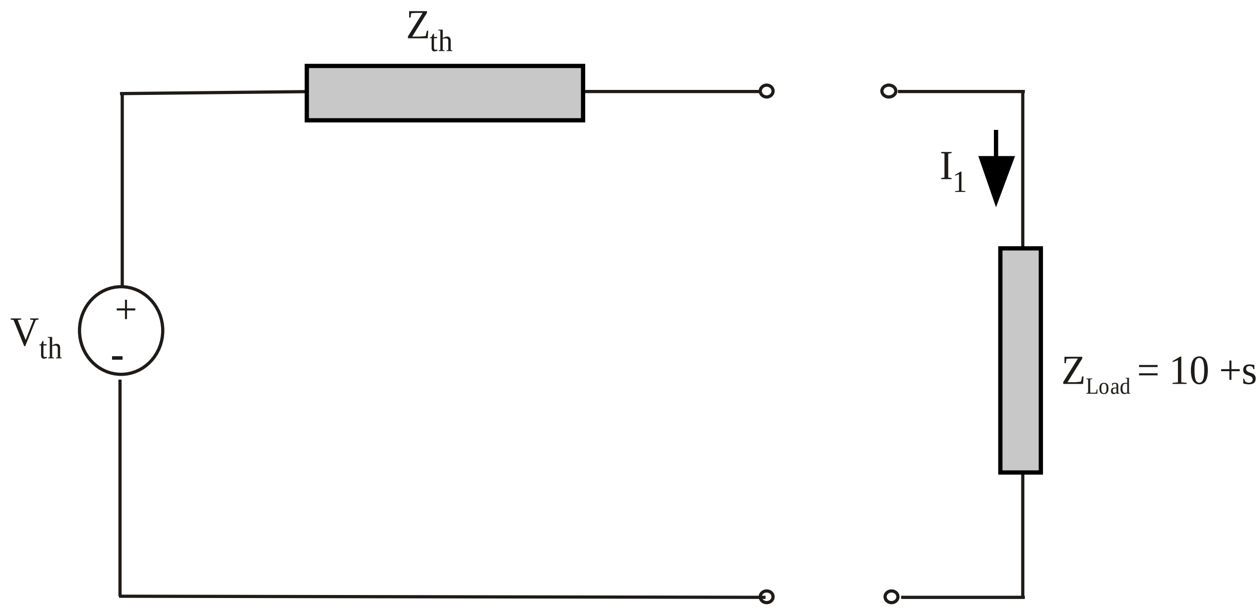
Transfer Function

Calculating the impedance



$$Z_{th} = \frac{10(s+10)}{(s+20)}$$

Transfer Function



$$I_1(s) = \frac{V_{th}(s)}{Z_{th}(s) + Z_{Load}} = \frac{1,000}{\frac{10(s+10)}{s+20} + (s+10)}$$

$$I_1(s) = \frac{1,000}{s\left(s^2 + 40s + 300\right)}$$

Expanding using partial fraction

$$I_1(s) = \frac{K_1}{s} + \frac{K_2}{(s+10)} + \frac{K_3}{(s+30)}$$

With K_1 = 3.33, K_2 = -5, K_3 = 1.67

The time-domain current $i_1(t)$ is:

$$i_1(t) = 3.33 - 5e^{-10t} + 1.67e^{-30t} \quad A$$

Octave & Decade

If $f_2 = 2f_1$, then f_2 is one octave above f_1

If $f_2 = 10f_1$, then f_2 is one decade above f_1

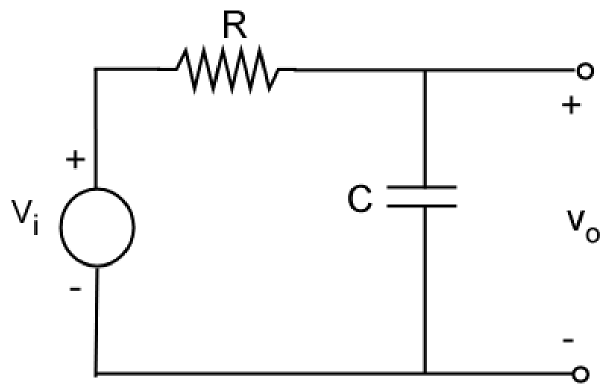
$$\#\,of\;\; octaves = \log_2 \frac{f_2}{f_1} = 3.32\log_{10}\frac{f_2}{f_1}$$

$$\#\,of\;\; decades = \log_{10}\frac{f_2}{f_1}$$

2 GHz is one octave above 1 GHz

10 GHz is one decade above 1 GHz

Low-Pass Circuit



In frequency domain:

$$V_o = \frac{V_i}{R + \frac{1}{j\omega C}} \cdot \frac{1}{j\omega C}$$

$$V_o = \frac{V_i}{1 + j\omega RC} \Rightarrow A_v = \frac{V_o}{V_i} = \frac{1}{1 + j\omega RC}$$

$$A_v = \frac{1}{1 + j\omega RC} = \frac{1}{1 + jf / f_o}$$

Low-Pass Circuit

$$f_o = \frac{1}{2\pi RC} = \frac{1}{2\pi\tau}$$

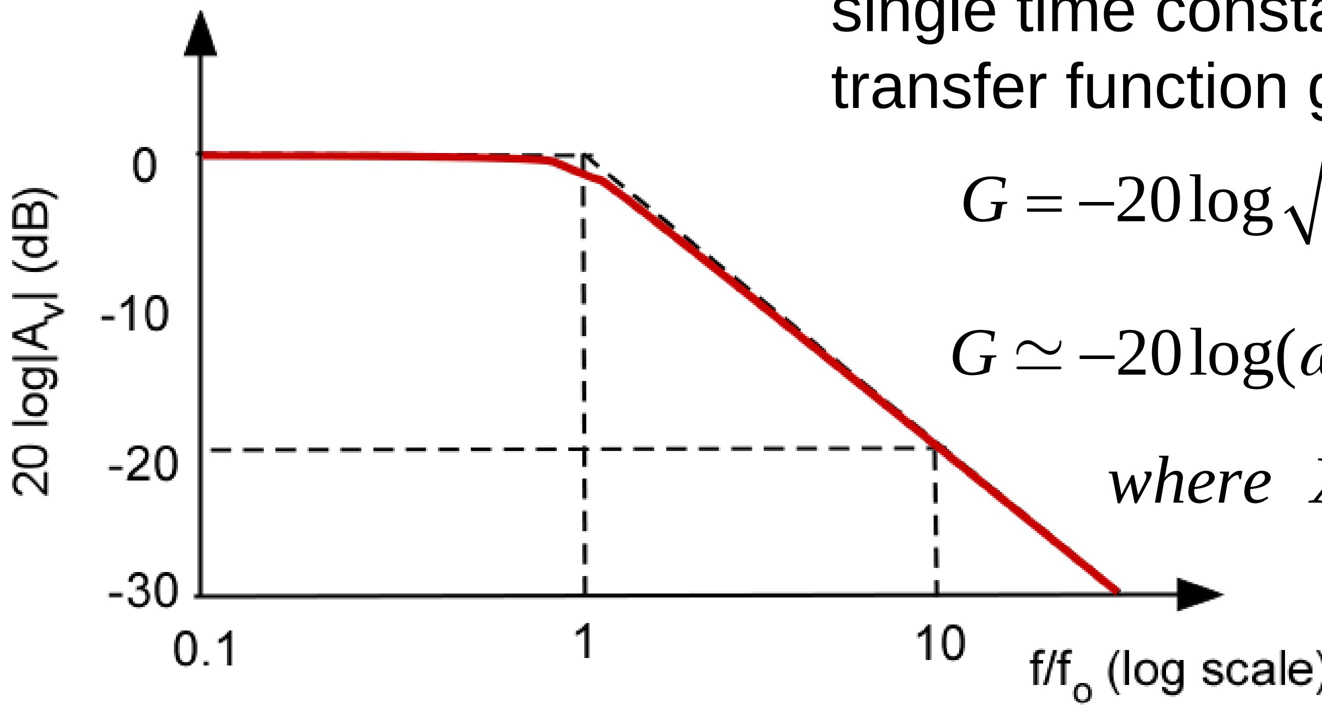
$$\tau = RC = \text{time constant}$$

At very high frequencies, the single time constant (STC) transfer function goes as

$$G = -20\log\sqrt{1+(\omega/\omega_o)^2}$$

$$G \simeq -20\log(\omega/\omega_o) = -20X$$

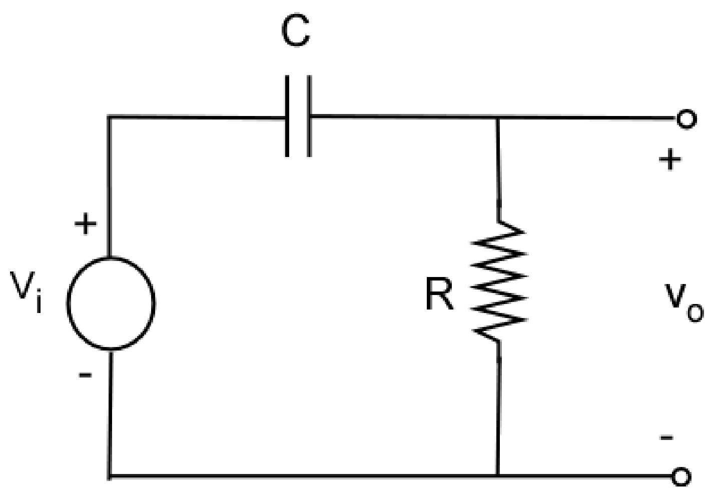
$$where \quad X = \log(\omega/\omega_o)$$

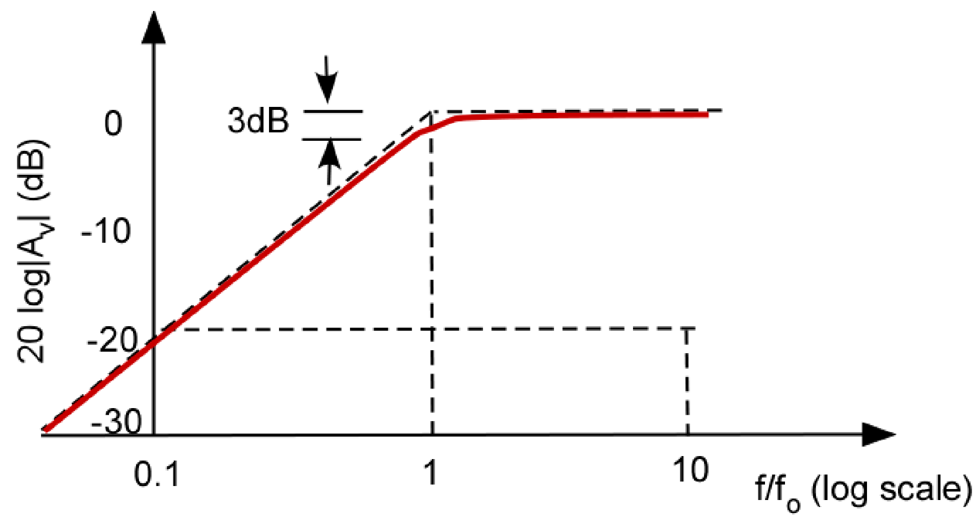


At high frequencies, slope of curve is –20 dB

$$if \quad X = 1\,(\omega = 10\omega_o),\ decrease\ is - 20\ dB \Rightarrow -20dB/decade$$

High-Pass Circuit





$$V_o = \frac{V_i R}{R + \frac{1}{j\omega C}} = \frac{V_i}{1 + \frac{1}{j\omega RC}}$$

$$A_v = \frac{V_o}{V_i} = \frac{1}{1 - j\frac{1}{2\pi fRC}} = \frac{1}{1 - jf_o / f}$$

Frequency Response

3-dB points are points where the magnitude is divided by $2^{1/2}$ (power is halved) $|1+j| = 2^{1/2}$

A_{dB}= 20log1.414 = 3 dB

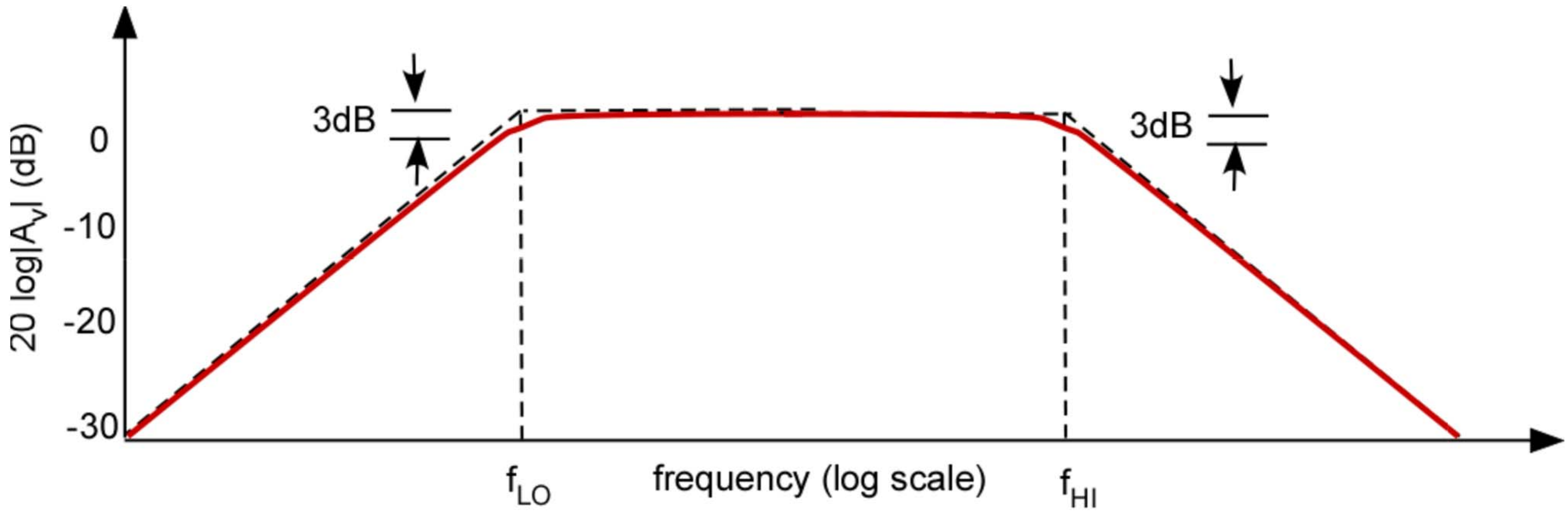
Amplifier has intrinsic gain A_o

Low-pass characteristics is: $$\frac{1}{1+jf/f_{hi}}$$

High-pass characteristics is: $$\frac{jf/f_{lo}}{1+jf/f_{lo}}$$

Overall gain $A(f)$ is $$A_o \cdot \frac{jf/f_{lo}}{1+jf/f_{lo}} \cdot \frac{1}{1+jf/f_{hi}}$$

Frequency Response



Overall gain *A(f)* is

$$A(f) = A_o \cdot \frac{jf / f_{lo}}{1 + jf / f_{lo}} \cdot \frac{1}{1 + jf / f_{hi}}$$