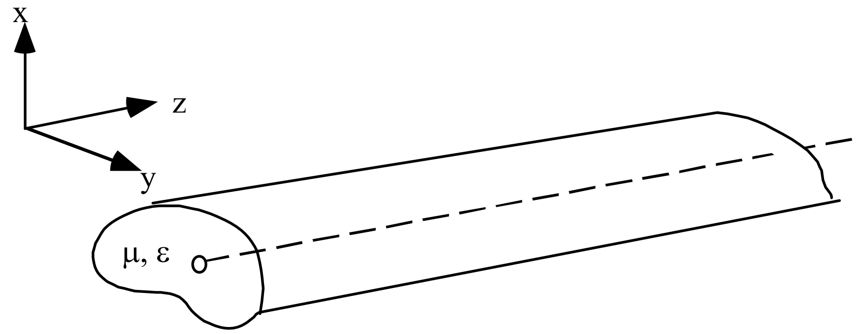
WAVEGUIDES



Maxwell's Equation

$$\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = \mathbf{0} \qquad \text{(A)}$$

$$\nabla^2\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H} = \mathbf{0} \qquad \text{(B)}$$

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set $E_Z = 0$. This defines the TE modes.

From $\nabla \times \mathbf{E} = -\mu \dfrac{\partial \mathbf{H}}{\partial t}$, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu\frac{\partial H_x}{\partial t} \Rightarrow +j\beta_z E_y = -j\omega\mu H_x \qquad (1)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu\frac{\partial H_y}{\partial t} \Rightarrow -j\beta_z E_x = -j\omega\mu H_y \qquad (2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu\frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad (3)$$

From $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$, we get

$$j\omega\varepsilon\mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega\varepsilon E_x \tag{4}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \Rightarrow -j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \tag{5}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \tag{6}$$

We want to express all quantities in terms of H_Z.

From (2), we have

$$H_y = \frac{\beta_z E_x}{\omega\mu} \tag{7}$$

in (4)

$$\frac{\partial H_z}{\partial y} + j\beta_z^2 \frac{E_x}{\omega\mu} = j\omega\varepsilon E_x \tag{8}$$

Solving for E_X

$$E_x = \frac{j\omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon} \frac{\partial H_z}{\partial y} \tag{9}$$

From (1)

$$H_x = \frac{-\beta_z E_y}{\omega\mu} \tag{10}$$

in (5)

$$j\frac{\beta_z^2 E_y}{\omega\mu} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \tag{11}$$

so that

$$E_y = \frac{-j\omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon} \frac{\partial H_z}{\partial x} \tag{12}$$

$$H_x = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial x} \tag{13}$$

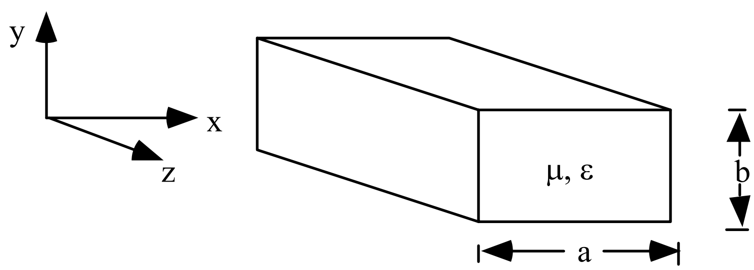
$$H_y = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial y} \tag{14}$$

$$E_Z = 0 \tag{15}$$

Combining solutions for E_X and E_Y into (3) gives

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[\beta_z^2 - \omega^2 \mu\varepsilon\right] H_z \tag{16}$$

RECTANGULAR WAVEGUIDES



If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

TE Modes

The general solution for TE modes with $E_Z=0$ is obtained from (16)

$$H_z = e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y} + De^{+j\beta_y y}\right] \tag{17}$$

$$E_y = \frac{\beta_x \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} e^{-j\beta_z z}\left[-Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y} + De^{+j\beta_y y}\right] \tag{18}$$

$$E_x = \frac{-\beta_y \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[-Ce^{-j\beta_y y} + De^{+j\beta_y y}\right] \tag{19}$$

At y=0, E_X=0 which leads to $C=D$

At x=0, E_Y=0 which leads to $A=B$

$$H_z = H_o e^{-j\beta_z z} \cos\beta_x x \cos\beta_y y \tag{20}$$

$$E_y = \frac{j\beta_x \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} H_o e^{-j\beta_z z} \sin\beta_x x \cos\beta_y y \tag{21}$$

$$E_x = \frac{-j\beta_y \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} H_o e^{-j\beta_z z} \cos\beta_x x \sin\beta_y y \tag{22}$$

At $x=a$, $E_y=0$; this leads to $\beta_x = \dfrac{m\pi}{a}$

At $y=b$, $E_x=0$; this leads to $\beta_y = \dfrac{n\pi}{b}$

The dispersion relation is obtained by placing (20) in (16)

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2 \mu\varepsilon \tag{23}$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \omega^2 \mu\varepsilon \tag{24}$$

and

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{25}$$

The guidance condition is

$$\omega^2 \mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{26}$$

or $f > f_c$ where f_c is the cutoff frequency of the TE_{mn} mode given by the relation

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{27}$$

The TE_{mn} mode will not propagate unless f is greater than f_c. Obviously, different modes will have different cutoff frequencies.

TM Modes

The transverse magnetic modes for a general waveguide are obtained by assuming H_Z =0. By duality with the TE modes, we have

$$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}=\left[\beta_z^2-\omega^2\mu\varepsilon\right]E_z \tag{28}$$

with general solution

$$E_z=e^{-j\beta_z z}\left[Ae^{-j\beta_x x}+Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y}+De^{+j\beta_y y}\right] \tag{29}$$

The boundary conditions are

At $x=0$, $E_Z=0$ which leads to $A=-B$

At $y=0$, $E_Z=0$ which leads to $C=-D$

At $x=a$, $E_Z=0$ which leads to $\beta_x=\frac{m\pi}{a}$

At $y=b$, $E_Z=0$ which leads to $\beta_y=\frac{n\pi}{b}$

so that the generating equation for the TM_{mn} modes is

$$E_z=E_o e^{-j\beta_z z}\sin\beta_x x\sin\beta_y y \tag{30}$$

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

For additional information on the field equations see **Rao (6th Edition), page 607, Table 9.1**.

There is no TE_{00} mode

There are no TM_{m0} or TM_{0n} modes

The first TE mode is the TE_{10} mode

The first TM mode is the TM_{11} mode

Impedance of a Waveguide

For a TE mode, we define the transverse impedance as

$$\eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta_z}$$

From the relationship for β_z and using

$$f_c^2 = \frac{1}{4\mu\varepsilon}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \tag{31}$$

we get

$$\eta_{gTE} = \frac{\eta}{\sqrt{1-\frac{f_c^2}{f^2}}} \tag{32}$$

where η is the intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ Analogously, for TM modes, it can be shown that

$$\eta_{gTM} = \eta\sqrt{1-\frac{f_c^2}{f^2}} \tag{33}$$

Power Flow in a Rectangular Waveguide (TE_{10})

The time-average Poynting vector for the TE_{10} mode in a rectangular waveguide is given by

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right] = \hat{\mathbf{z}}\frac{|E_o|^2}{2}\frac{\beta_z}{\omega\mu}\sin^2\frac{\pi x}{a} \tag{34}$$

$$\langle Power \rangle = \int_0^a\int_0^b \frac{|E_o|^2}{2}\frac{\beta_z}{\omega\mu}\sin^2\frac{\pi x}{a}dxdy \tag{35}$$

$$\langle Power \rangle = \frac{|E_o|^2}{4}\frac{\beta_z ab}{\omega\mu} = \frac{|E_o|^2}{4}\frac{ab}{\eta_{gTE_{10}}} \tag{36}$$

Therefore, the time-average power flow in a waveguide is proportional to its cross-section area.