ECE451 Homework 2 – Solutions

Problem 1

Mode	f_c (GHz)	θ (deg)	v_{pz} (m/s)	v_g (m/s)
TEM	0	90	0.1×10^9	0.1×10^9
TE_1	2	66.42	0.1091×10^9	0.0916×10^9
TM_1	2	66.42	0.1091×10^9	0.0916×10^9
TE_2	4	86.86	0.16×10^9	0.06×10^9
TM_2	4	36.86	0.16×10^9	0.06×10^9

$$\lambda = \frac{v_p}{f}, \quad f_c = \frac{m}{2a\sqrt{\mu\varepsilon}}, \quad \theta = \cos^{-1}\frac{\lambda}{\lambda_c}$$

$$\lambda_c = \frac{v_p}{f_c} = \frac{2a}{m}, \quad \lambda_g = \frac{v_{pz}}{f}, v_{pz} = \frac{v_p}{\sqrt{1-\frac{f_c^2}{f^2}}}, \quad v_g = v_p\sqrt{1-\frac{f_c^2}{f^2}}$$

Problem 2

$$Set\ \frac{c}{a} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

$$\frac{c^2}{a^2} = \frac{c^2}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

$$\frac{1}{a^2} - \frac{1}{4a^2} = \frac{1}{4b^2} \Rightarrow b = \frac{a}{\sqrt{3}} = \frac{1}{\sqrt{3}} = 0.57\ cm$$

Problem 3

Solution:

(a) For the TE_{11} mode, we have

$$f_{cTE_{11}} = \frac{1.8412c}{2\pi a} = \frac{1.8412 \times 3 \times 10^8}{\pi \times 2.383 \times 10^{-2}} \simeq 7.38GHz$$

For the TM_{01} and TE_{21} modes, we have

$$f_{cTM_{01}} = \frac{2.4049}{1.8412} f_{cTE_{11}} \simeq 9.64 \text{ GHz}$$

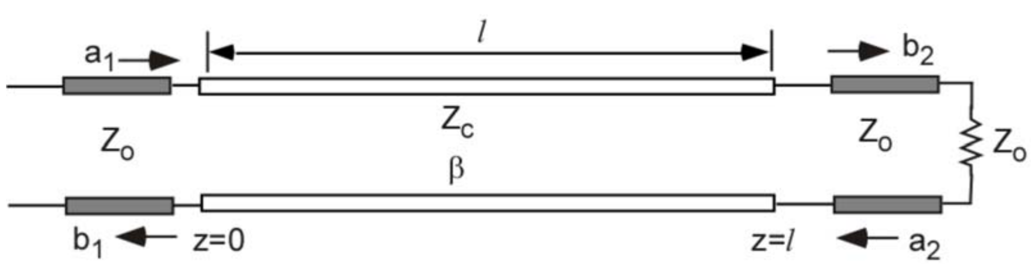
$$f_{cTE_{21}} = \frac{3.0542}{1.8412} f_{cTE_{11}} \simeq 12.2 \text{ GHz}$$

(b) At 10 GHz, the only mode that will propagate through this guide are the TE_{11} and TM_{01}

(c)The frequency range over which only the dominant TE_{11} mode propagates along the guide can be found as

Frequency range f_{cTM01}-f_{cTE11} = 9.64-7.38 = 2.26 GHz.

Problem 4



(a) Calculating S_{11}

At the load $(z=l)$, the reflection coefficient is: $\Gamma_R = \frac{Z_o - Z_c}{Z_o + Z_c} = -\Gamma$

At the input of the transmission line *(z=0)*, the reflection coefficient is Γ_{in}

$$\Gamma_{in} = \Gamma_R e^{-2j\beta l} = -\Gamma X^2 \text{ with } X = e^{-j\beta l}$$

The input impedance is:

$$Z_{in} = Z_c \left[\frac{1-\Gamma X^2}{1+\Gamma X^2} \right]$$

S_{11} is the reflection coefficient at the input with respect to reference line (with characteristic impedance Z_o)

$$S_{11} = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{Z_c \left[\frac{1-\Gamma X^2}{1+\Gamma X^2} \right] - Z_o}{Z_c \left[\frac{1-\Gamma X^2}{1+\Gamma X^2} \right] - Z_o} = \frac{-\Gamma X^2 \left[Z_c + Z_o \right] + Z_c - Z_o}{-\Gamma X^2 \left[Z_c - Z_o \right] + Z_c + Z_o}$$

$$S_{11} = \frac{\Gamma - \Gamma X^2}{1 - \Gamma^2 X^2} = \frac{(1 - X^2)\Gamma}{1 - \Gamma^2 X^2}$$

(b) Calculating S_{21}

for a test line, we have

$$V_t(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}$$

$$I_t(z) = \frac{1}{Z_c} \left[V_+ e^{-j\beta z} - V_- e^{+j\beta z} \right]$$

Let us define:

$$\Gamma = \frac{Z_c - Z_o}{Z_c + Z_o}$$

$$X = e^{-j\beta l}$$

We also define

$$A_1 = a_1 \sqrt{Z_o}\,, \quad B_1 = b_1 \sqrt{Z_o}$$

By definition S_{21} is given by:

$$S_{21} = \frac{b_2}{a_1} = \frac{V_t(l)}{a_1 \sqrt{Z_o}} = \frac{V_t(l)}{A_1}$$

$$V_t(z) = V_+ e^{-j\beta z} \left[1 + \frac{V_-}{V_+} e^{+2j\beta z} \right]$$

$$\frac{V_-}{V_+} = \Gamma_t(0) \Rightarrow V_t(z) = V_+ e^{-j\beta z}\left[1 + \Gamma_t(0) e^{+2j\beta z}\right]$$

$$\Gamma_t(0) = \Gamma_t(l) e^{-2j\beta l} = \Gamma_t(l) X^2 = -\Gamma X^2$$

so that $V_t(l) = V_+ e^{-j\beta l}\left[1 + \Gamma_t(0) e^{+2j\beta l}\right] = V_+ X\left[1 - \Gamma X^2 X^{-2}\right] = V_+ X\left[1 - \Gamma\right]$ (1)

We now need to find *V+*

At the junction between the reference and the test line, the voltage and the current must be continuous. This gives:

$$A_1 + B_1 = V_+ + V_-$$

and

$$\frac{1}{Z_o}\left[A_1 - B_1\right] = \frac{1}{Z_c}\left[V_+ - V_-\right]$$

from which

$$A_1 + B_1 = V_+\left[1 + \Gamma_t(0)\right] = V_+\left[1 - \Gamma X^2\right] \quad (2)$$

$$A_1 - B_1 = \frac{Z_o}{Z_c} V_+\left[1 - \Gamma_t(0)\right] = \frac{Z_o}{Z_c} V_+\left[1 + \Gamma X^2\right] \quad (3)$$

Adding (2) and (3)

$$2A_1 = V_+\left[1 - \Gamma X^2 + \frac{Z_o}{Z_c}\left(1 + \Gamma X^2\right)\right]$$

Extracting *V+, we get:*

$$V_+ = \frac{2A_1}{\left[1 - \Gamma X^2 + \frac{Z_o}{Z_c}\left(1 + \Gamma X^2\right)\right]} = \frac{2A_1}{\left[1 - \Gamma X^2 \Gamma\right]\left(1 + \frac{Z_o}{Z_c}\right)} = \frac{2A_1}{\left[1 + \frac{Z_o}{Z_c}\right]}\frac{1}{1 - \Gamma^2 X^2} \quad (4)$$

Since $\frac{2}{\left[1 + \frac{Z_o}{Z_c}\right]} = 1 + \Gamma$, substituting in (4) gives $V_+ = \frac{\left[1 + \Gamma\right] A_1}{1 - \Gamma^2 X^2}$

Substituting for *V+* in (1) gives $V_t(l) = \frac{A_1\left[1 + \Gamma\right]\left[1 - \Gamma\right] X}{1 - \Gamma^2 X^2} = \frac{A_1\left[1 - \Gamma^2\right] X}{1 - \Gamma^2 X^2}$

so that $S_{21} = \frac{V_t(l)}{A_1} = \frac{\left(1-\Gamma^2\right)X}{1-\Gamma^2 X^2}$

In conclusion:

$$S_{11} = \frac{\left(1-X^2\right)\Gamma}{1-\Gamma^2 X^2} \text{ and } S_{21} = \frac{\left(1-\Gamma^2\right)X}{1-\Gamma^2 X^2}$$

Problem 5

(a) The network is reciprocal since $S_{ij}=S_{ji}$ for all i, j

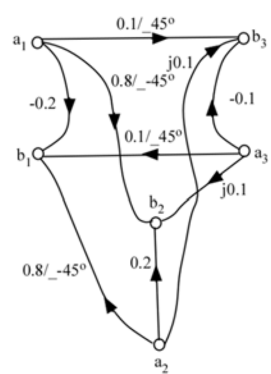
(b) Remember that S_{ij}^2 is the fraction of power leaving port j for a unit of power input to port i. The criterion for a network with N ports to be lossless is $\sum_{j=1}^{N} \left|S_{ji}\right|^2 = 1$ for all i

(c)

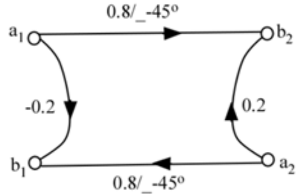
Port 1: $\left|S_{11}\right|^2 + \left|S_{21}\right|^2 + \left|S_{31}\right|^2 = 0.04 + 0.64 + 0.01 = 0.69 < 1$

The 3-port is lossy

(d)

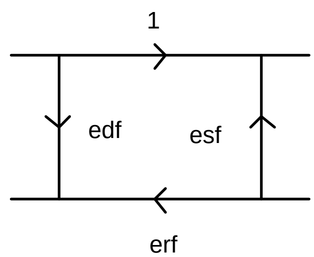


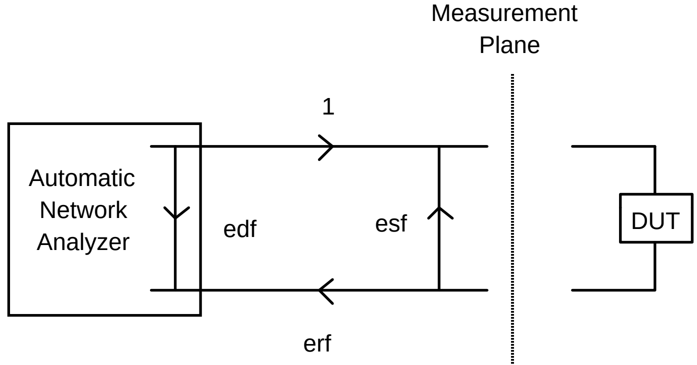
(e) With a 50 Ω load at port 3, there is no input at a_3. So, the SFG in (d) simplifies to



$$S : \begin{bmatrix} -0.2 & 0.8\angle-45° \\ 0.8\angle-45° & 0.2 \end{bmatrix}$$

Problem 6





(a)

$$\text{(b)}\ \ S_{11m} = E_{DF} + \frac{E_{RF} S_{11a}}{1 - S_{11a} E_{SF}}$$

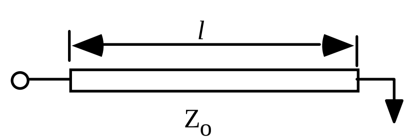
It is best to solve part (b) first, and then use part (a) as a special case.

$$\left.\begin{array}{c} \textit{Matched termination} \\ \textit{Offset short} \\ \textit{Shielded open} \end{array}\right\} \textit{Cal standards}$$

(1) Matched termination $\Rightarrow S11a = 0$

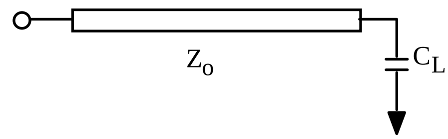
$$S_{11m}(match) = A = E_{DF} \quad (1)$$

(2) Offset short $\Rightarrow S_{11a} = e^{j\theta} = x$



$$\theta = \pi\left(1 - \frac{4l}{\lambda}\right)$$

$$S_{11m}(\textit{offset short}) = E_{DF} + \frac{E_{RF} x}{1 - xE_{SF}} = B \qquad (2)$$



$$S_{11a} = e^{j\beta} = y$$

$$\beta = -2\tan^{-1}\left(\omega C_L Z_o\right)$$

$$S_{11m}(shielded\ open) = E_{DF} + \frac{E_{RF}y}{1 - yE_{SF}} = C \qquad (3)$$

Combining (2) and (3) with (1), we get

$$\begin{bmatrix} x(B-A) & x \\ y(C-A) & y \end{bmatrix} \begin{bmatrix} E_{SF} \\ E_{RF} \end{bmatrix} = \begin{bmatrix} B-A \\ C-A \end{bmatrix}$$

From which

$$E_{SF} = \frac{y(B-A) - x(C-A)}{xy(B-C)}, \quad E_{RF} = \frac{(B-A)(C-A)(x-y)}{xy(B-C)}, \quad E_{DF} = A$$

For part (a) set $C_L = 0 \Rightarrow \beta = 0 \Rightarrow S_{11a}(open) = +1 \Rightarrow y = +1$

$$E_{DF} = A,\ E_{SF} = \frac{(B-A) - x(C-A)}{x(B-C)},\ E_{RF} = \frac{(B-A)(C-A)(x-1)}{x(B-C)}$$