ECE 451
Advanced Microwave Measurements

The Smith Chart

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Derivation of the Smith Chart

The relationship between impedance and reflection coefficient is given by:

\[ Z(z) = Z_0 \left[ \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right] \]

where \( Z_0 \) is the characteristic impedance of the system. The normalized impedance is

\[ Z_n(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = \frac{1 + \Gamma}{1 - \Gamma} \]

The reflection coefficient and the normalized impedance have the form:

\[ \Gamma = \Gamma_r + j\Gamma_i \quad \text{and} \quad Z_n = r + jx \]
Derivation of the Smith Chart

Therefore

\[ r + jx = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} = \left[ (1 + \Gamma_r) + j\Gamma_i \right] \left[ (1 - \Gamma_r) + j\Gamma_i \right] \]

\[ \frac{(1 - \Gamma_r)^2 + \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \]

Separating real and imaginary components,

\[ r + jx = \frac{1 - \Gamma_r^2 + j\Gamma_i(1 + \Gamma_r) + j\Gamma_i(1 - \Gamma_r) - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \]

Isolating the real part from both sides

\[ r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \]
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Multiplying through by the denominator,

\[ r \left[ 1 + \Gamma_r^2 - 2\Gamma_r + \Gamma_i^2 \right] = 1 - \Gamma_r^2 - \Gamma_i^2 \]

\[ \Gamma_r^2 (r+1) + \Gamma_i^2 (r+1) - 2r\Gamma_r = 1-r \]

\[ \Gamma_r^2 + \Gamma_i^2 - \frac{2r\Gamma_r}{1+r} + \frac{r^2}{(1+r)^2} = \frac{1-r}{1+r} + \frac{r^2}{(1+r)^2} \]

Completing the square

\[ \Gamma_r^2 + \Gamma_i^2 - \frac{2r\Gamma_r}{1+r} = \frac{1-r}{1+r} \quad \text{or} \quad \left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2} \]
Derivation of the Smith Chart

\[
\left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2}
\]

This is the equation of a circle centered at \( \left( \frac{r}{1+r}, 0 \right) \) and of radius \( \frac{1}{1+r} \).

Equating the imaginary parts gives

\[
x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}
\]

\[
x\left[ 1 + \Gamma_r^2 - 2\Gamma_r + \Gamma_i^2 \right] = 2\Gamma_i \quad \text{or} \quad \Gamma_r^2 x - 2x\Gamma_r + x\Gamma_i^2 - 2\Gamma_i = -x
\]
Derivation of the Smith Chart

\[ \Gamma_r^2 - 2\Gamma_r + 1 + \Gamma_i^2 - \frac{2\Gamma_i}{x} + \frac{1}{x^2} = \frac{1}{x^2} - 1 + 1 \]

\[ (\Gamma_r - 1)^2 + \left( \Gamma_i - \frac{1}{x} \right)^2 = \frac{1}{x^2} \]

This is the equation of a circle centered at

\[ \left( 1, \frac{1}{x} \right) \text{ of radius } \frac{1}{x} \]
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The reflection coefficient is given by

$$\Gamma = \frac{Z_n - 1}{Z_n + 1} = \frac{r - 1 + jx}{r + 1 + jx}$$

We also have

$$|\Gamma| = \left[\frac{(r - 1)^2 + x^2}{(r + 1)^2 + x^2}\right]^{1/2} \leq 1$$

$$Z_n = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

$$y = \frac{l}{Z_n} = \frac{l - \Gamma(z)}{l + \Gamma(z)}$$

Thus, going from normalized impedance to normalized admittance corresponds to a 180 degree shift.
The Smith Chart

3 ways to move on the Smith chart

- Constant SWR circle ➞ displacement along TL
- Constant resistance (conductance) circle ➞ addition of reactance (susceptance)
- Constant reactance (susceptance) arc ➞ addition of resistance (conductance)
The Smith Chart
Smith Chart Example

FIND:

1. Reflection coefficient at load

\[ z_R = 0.3 - j0.4 \Rightarrow \Gamma_R = 0.6e^{j227^\circ} \]

2. SWR on the line

SWR=4.0

3. \( d_{\text{min}} \)

\[ d_{\text{min}} = (0.5 - 0.435)\lambda = 0.065\lambda \]
4. Line impedance at 0.05\( \lambda \) to the left
\[ 50(0.26 - j0.09) = 13 - j4.5\Omega \]

5. Line admittance at 0.05\( l \)
\[ \left(3.5 + j1.2\right)/50 = 0.068 + j0.025\, \Omega \]

6. Location nearest to load where Real\( [y] \)=1
\[ 0.14\lambda = 0.325\lambda - j0.185\lambda = 0.14\lambda \]
Smith Chart Example