

ECE 451

Advanced Microwave Measurements

Transmission Lines

Jose E. Schutt-Aine
Electrical & Computer Engineering
University of Illinois
jesa@illinois.edu

Maxwell's Equations

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

Faraday's Law of Induction

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

Ampère's Law

$$\nabla \cdot D = \rho$$

Gauss' Law for electric field

$$\nabla \cdot B = 0$$

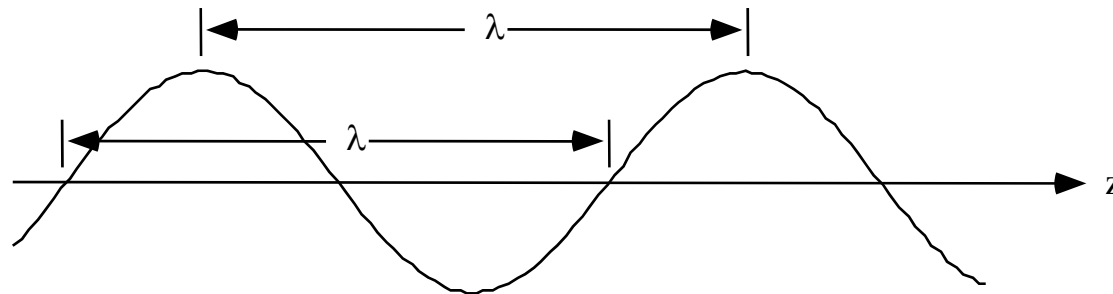
Gauss' Law for magnetic field

Constitutive Relations

$$B = \mu H$$

$$D = \epsilon E$$

Why Transmission Line?



Wavelength : λ

$$\lambda = \frac{\text{propagation velocity}}{\text{frequency}}$$

Why Transmission Line?

In Free Space

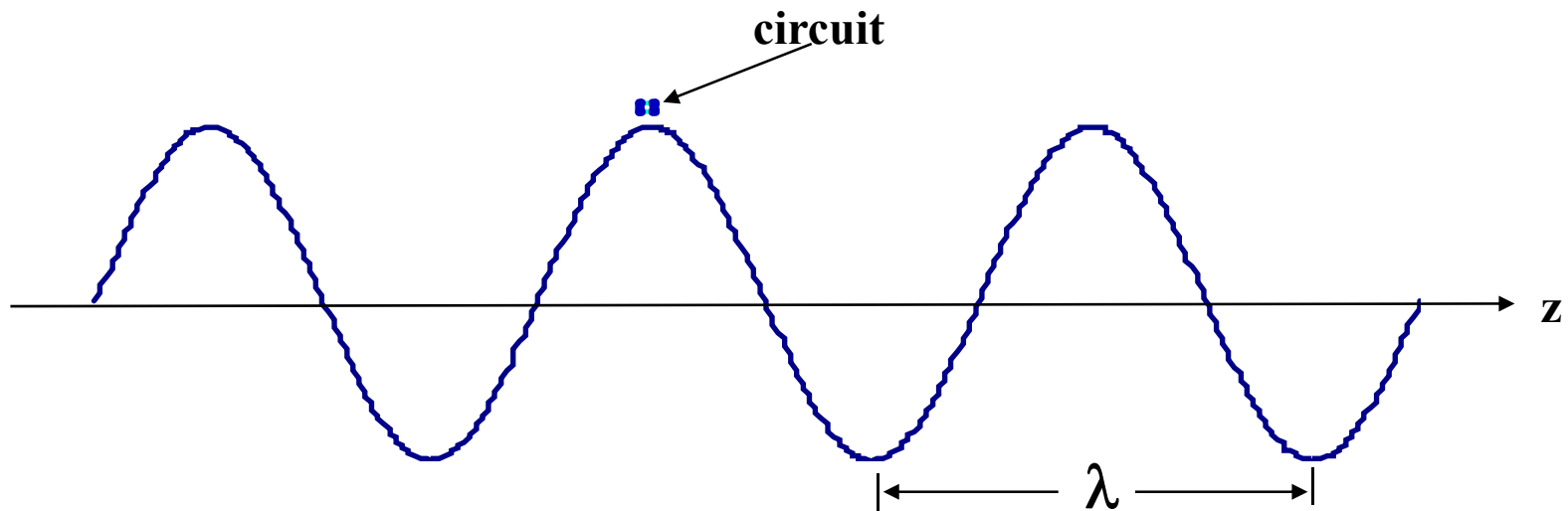
At 10 KHz : $\lambda = 30$ km

At 10 GHz : $\lambda = 3$ cm

Transmission line behavior is prevalent when the structural dimensions of the circuits are comparable to the wavelength.

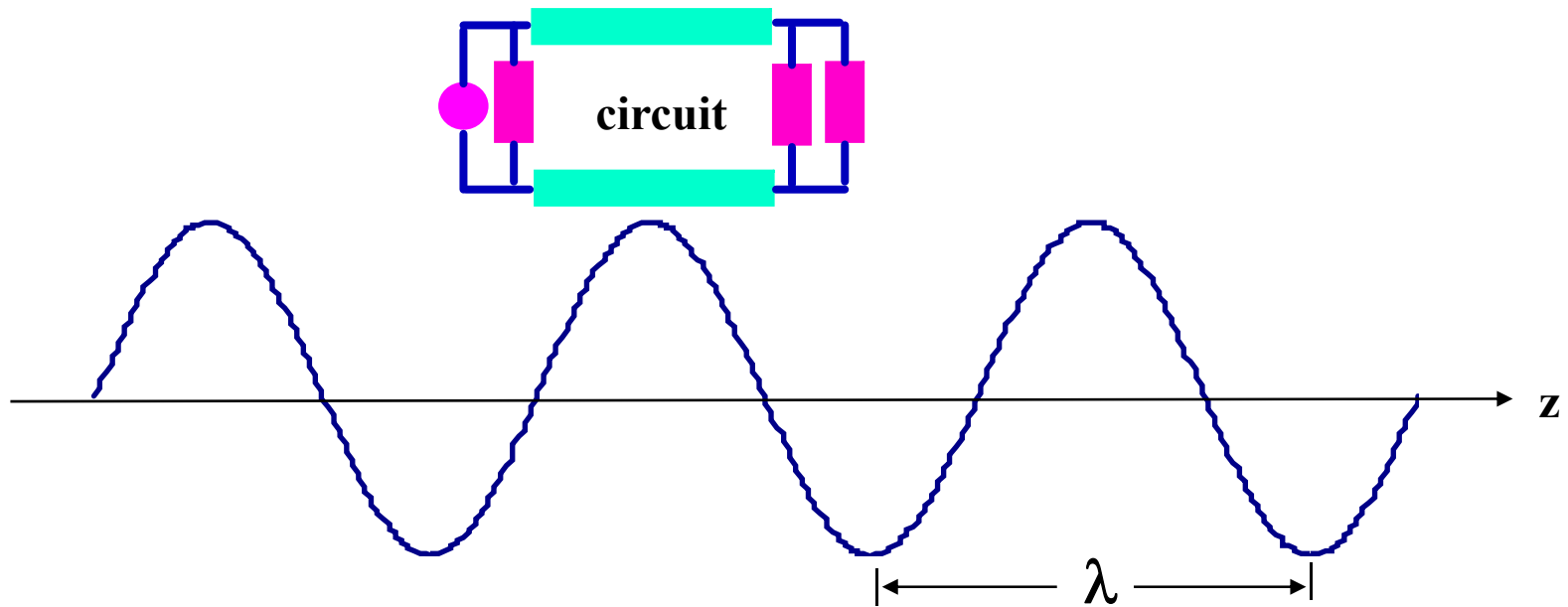
Justification for Transmission Line

Let d be the largest dimension of a circuit



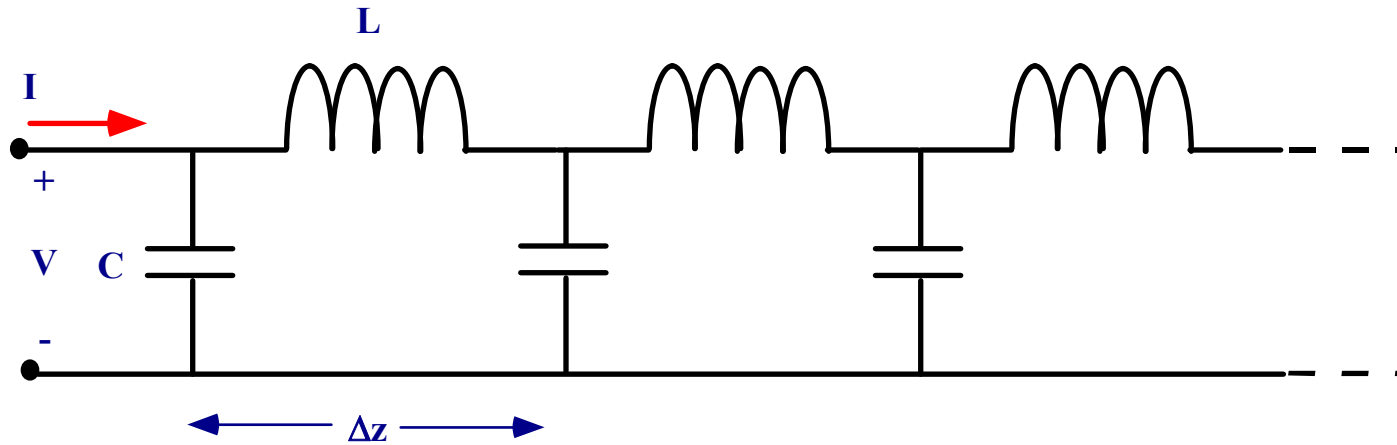
If $d \ll \lambda$, a lumped model for the circuit can be used

Justification for Transmission Line



If $d \approx \lambda$, or $d > \lambda$ then use transmission line model

Telegraphers' Equations



L: Inductance per unit length.

C: Capacitance per unit length.

$$-\frac{\partial V}{\partial z} = L \frac{\partial I}{\partial t}$$

Assume
time-harmonic
dependence

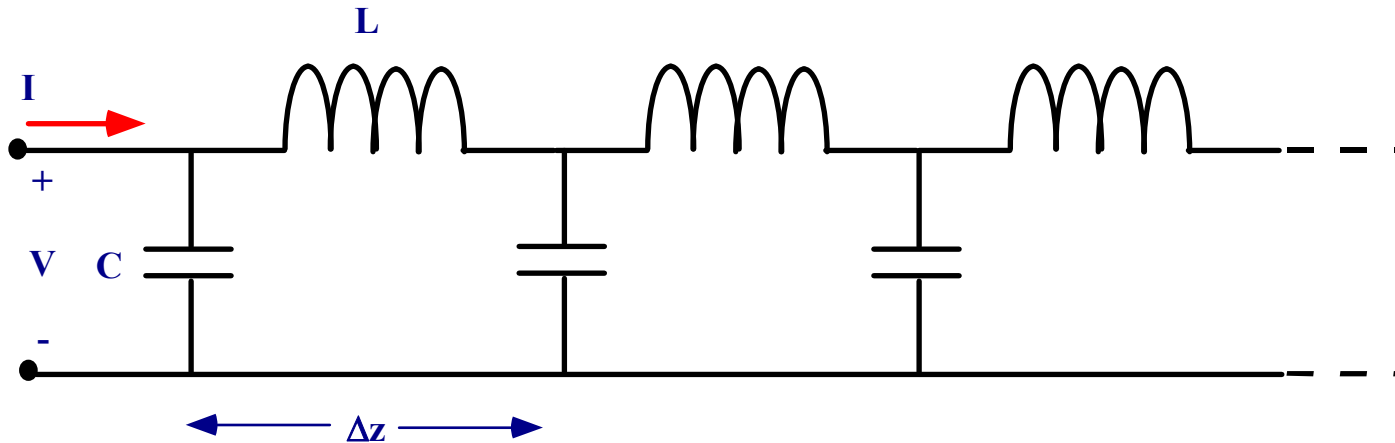
$$-\frac{\partial V}{\partial z} = j\omega LI$$

$$\frac{\partial I}{\partial z} = C \frac{\partial V}{\partial t}$$

$$V, I \sim e^{j\omega t}$$

$$\frac{\partial I}{\partial z} = j\omega CV$$

TL Solutions

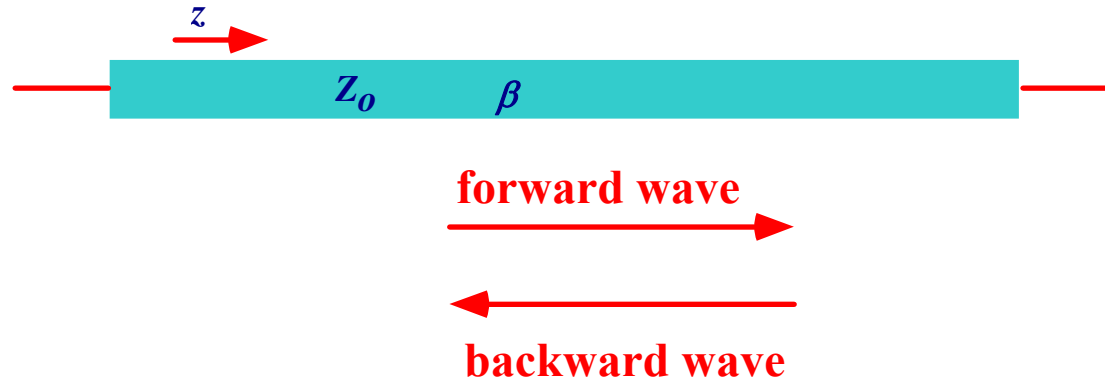


$$-\frac{\partial}{\partial z} \left(\frac{\partial V}{\partial z} \right) = -\frac{\partial^2 V}{\partial z^2} = j\omega L \frac{\partial I}{\partial z} = -j\omega L j\omega C V \quad \longrightarrow \quad \frac{\partial^2 V}{\partial z^2} = -\omega^2 L C V$$

$$-\frac{\partial}{\partial z} \left(\frac{\partial I}{\partial z} \right) = -\frac{\partial^2 I}{\partial z^2} = j\omega C \frac{\partial V}{\partial z} = -j\omega L j\omega C I \quad \longrightarrow \quad \frac{\partial^2 I}{\partial z^2} = -\omega^2 C L I$$

TL Solutions

(Frequency Domain)



$$\beta = \omega\sqrt{LC}$$

$$V(z) = \overbrace{V_+ e^{-j\beta z}}^{\text{Forward Wave}} + \overbrace{V_- e^{+j\beta z}}^{\text{Backward Wave}}$$

$$Z_o = \sqrt{\frac{L}{C}}$$

$$I(z) = \underbrace{\frac{V_+}{Z_o} e^{-j\beta z}}_{\text{Forward Wave}} - \underbrace{\frac{V_-}{Z_o} e^{+j\beta z}}_{\text{Backward Wave}}$$

TL Solutions

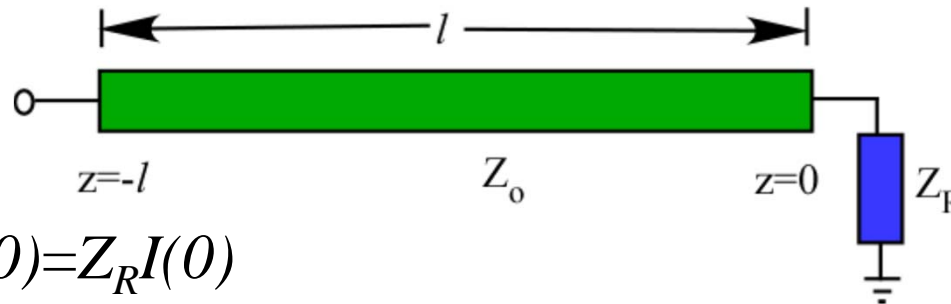
Propagation constant $\beta = \omega\sqrt{LC}$ **Propagation velocity** $v = \frac{1}{\sqrt{LC}}$

Characteristic impedance $Z_o = \sqrt{\frac{L}{C}}$ **Wavelength** $\lambda = \frac{v}{f}$

$$V(z,t) = \overbrace{V_+ \cos(\omega t - \beta z)}^{\text{Forward Wave}} + \overbrace{V_- \cos(\omega t + \beta z)}^{\text{Backward Wave}}$$

$$I(z,t) = \underbrace{\frac{V_+}{Z_o} \cos(\omega t - \beta z)}_{\text{Forward Wave}} - \underbrace{\frac{V_-}{Z_o} \cos(\omega t + \beta z)}_{\text{Backward Wave}}$$

Reflection Coefficient



At $z=0$, we have $V(0)=Z_R I(0)$

But from the TL equations:

$$\begin{aligned} V(0) &= V_+ + V_- \\ I(0) &= \frac{V_+}{Z_0} - \frac{V_-}{Z_0} \end{aligned} \quad \frac{Z_R}{Z_0} (V_+ - V_-) = V_+ + V_-$$

Which gives $V_- = \Gamma_R V_+$

where $\Gamma_R = \frac{Z_R - Z_0}{Z_R + Z_0}$ is the load reflection coefficient

Reflection Coefficient

- If $Z_R = Z_o$, $\Gamma_R = 0$, no reflection, the line is matched
- If $Z_R = 0$, short circuit at the load, $\Gamma_R = -1$
- If $Z_R \rightarrow \text{inf}$, open circuit at the load, $\Gamma_R = +1$

V and I can be written in terms of Γ_R

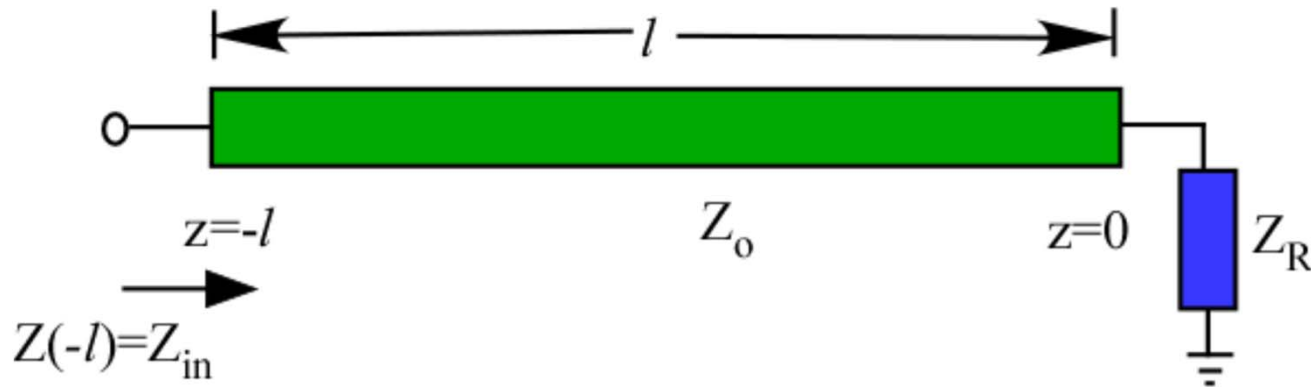
$$V(z) = V_+ \left[e^{-j\beta z} + \Gamma_R e^{+j\beta z} \right]$$

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

$$I(z) = \frac{V_+}{Z_o} \left[e^{-j\beta z} - \Gamma_R e^{+j\beta z} \right]$$

$$I(z) = \frac{V_+ e^{-j\beta z}}{Z_o} \left[1 - \Gamma_R e^{+2j\beta z} \right]$$

Generalized Impedance

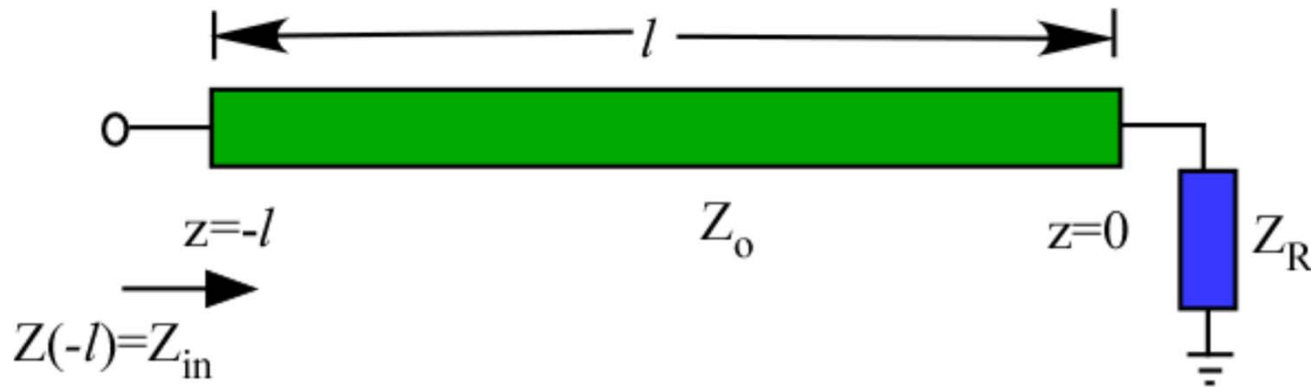


$$Z(z) = \frac{V(z)}{I(z)} = Z_o \left[\frac{e^{-j\beta z} + \Gamma_R e^{+j\beta z}}{e^{-j\beta z} - \Gamma_R e^{+j\beta z}} \right]$$

$$Z(-l) = Z_o \left[\frac{Z_R + jZ_o \tan \beta l}{Z_o + jZ_R \tan \beta l} \right]$$

Impedance
transformation
equation

Generalized Impedance



- Short circuit $Z_R = 0$, line appears inductive for $0 < l < \lambda/2$

$$Z(-l) = jZ_o \tan \beta l$$

- Open circuit $Z_R \rightarrow \text{inf}$, line appears capacitive for $0 < l < \lambda/2$

$$Z(-l) = \frac{Z_o}{j \tan \beta l}$$

- If $l = \lambda/4$, the line is a quarter-wave transformer

$$Z(-l) = \frac{Z_o^2}{Z_R}$$

Generalized Reflection Coefficient

$$\Gamma(z) = \frac{\text{Backward traveling wave at } z}{\text{Forward traveling wave at } z} = \frac{V_b(z)}{V_f(z)}$$

$$\Gamma(z) = \frac{V_- e^{+j\beta z}}{V_+ e^{-j\beta z}} = \frac{V_-}{V_+} e^{+2j\beta z} = \Gamma_R e^{+2j\beta z}$$

**Reflection coefficient
transformation equation**



$$\Gamma(-l) = \Gamma_R e^{-2j\beta l}$$

$$Z(z) = Z_o \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

$$\Gamma(z) = \frac{Z(z) - Z_o}{Z(z) + Z_o}$$

Voltage Standing Wave Ratio (VSWR)

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

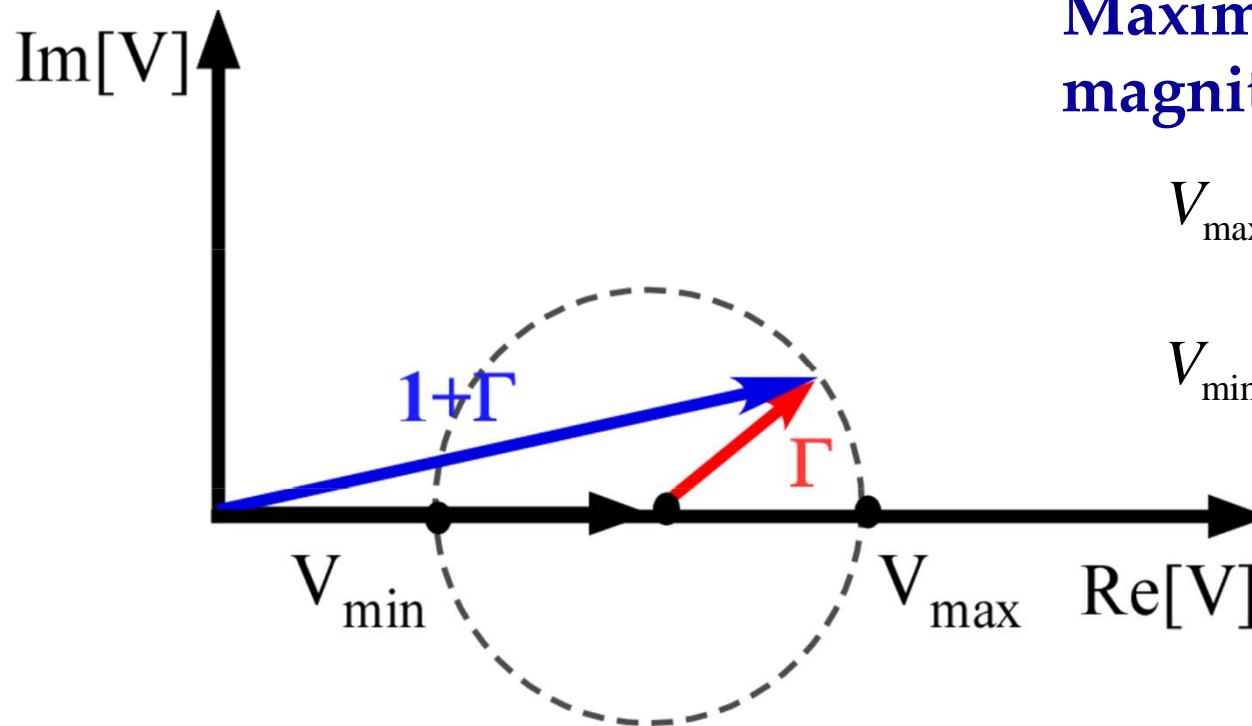
We follow the magnitude of the voltage along the TL

$$|V(z)| = |V_+ e^{-j\beta z}| |1 + \Gamma_R e^{+2j\beta z}| = |V_+| |1 + \Gamma_R e^{+2j\beta z}|$$

Maximum and minimum magnitudes given by

$$V_{\max} = [1 + |\Gamma_R|]$$

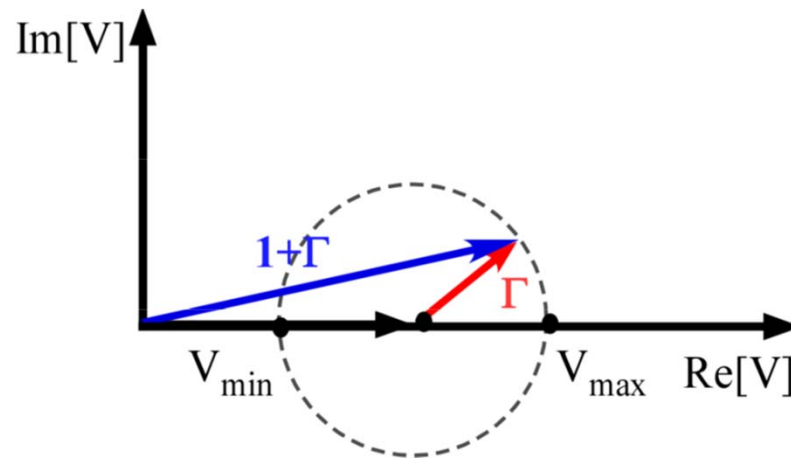
$$V_{\min} = [1 - |\Gamma_R|]$$



Voltage Standing Wave Ratio (VSWR)

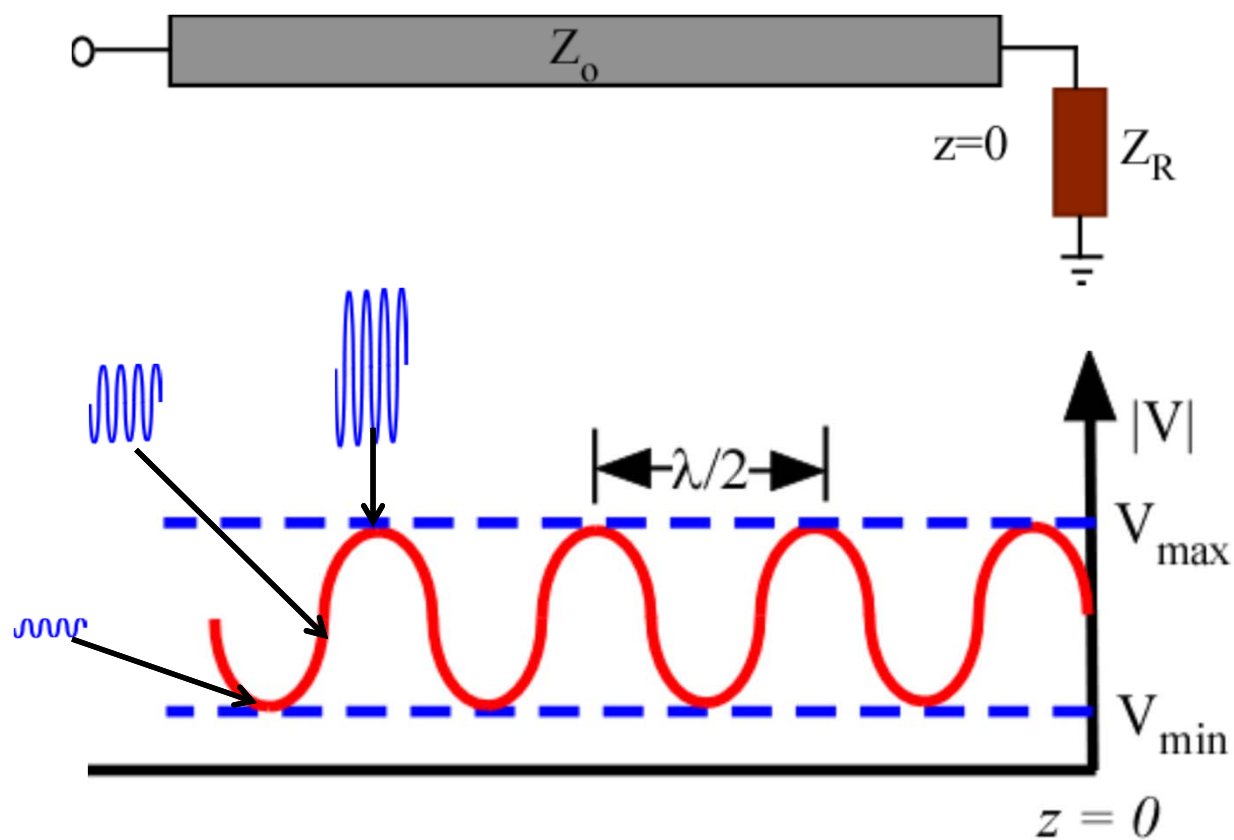
Define Voltage Standing Wave Ratio as:

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}$$



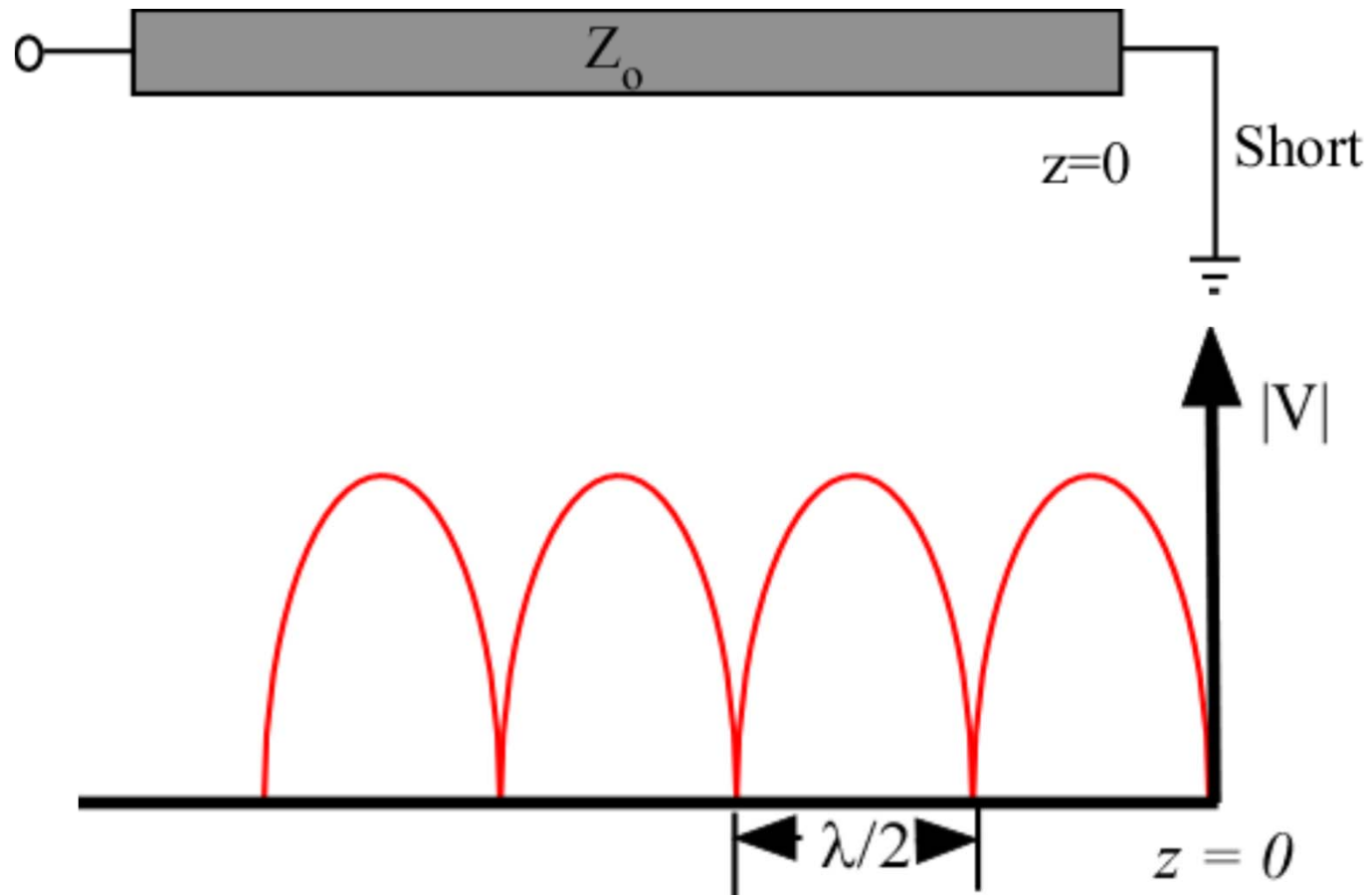
It is a measure of the interaction between forward and backward waves

VSWR – Arbitrary Load



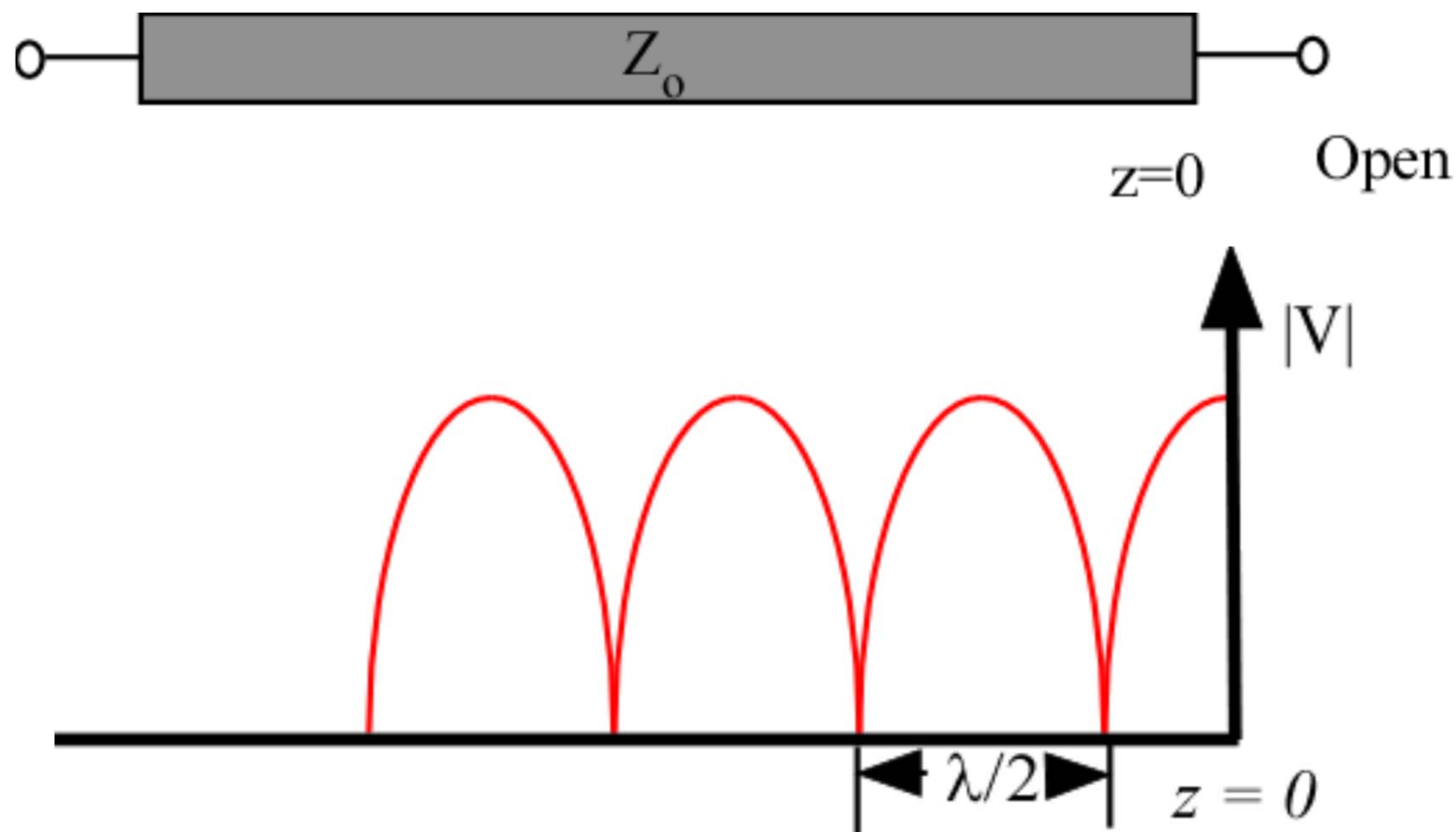
Shows variation of amplitude along line

VSWR – For Short Circuit Load



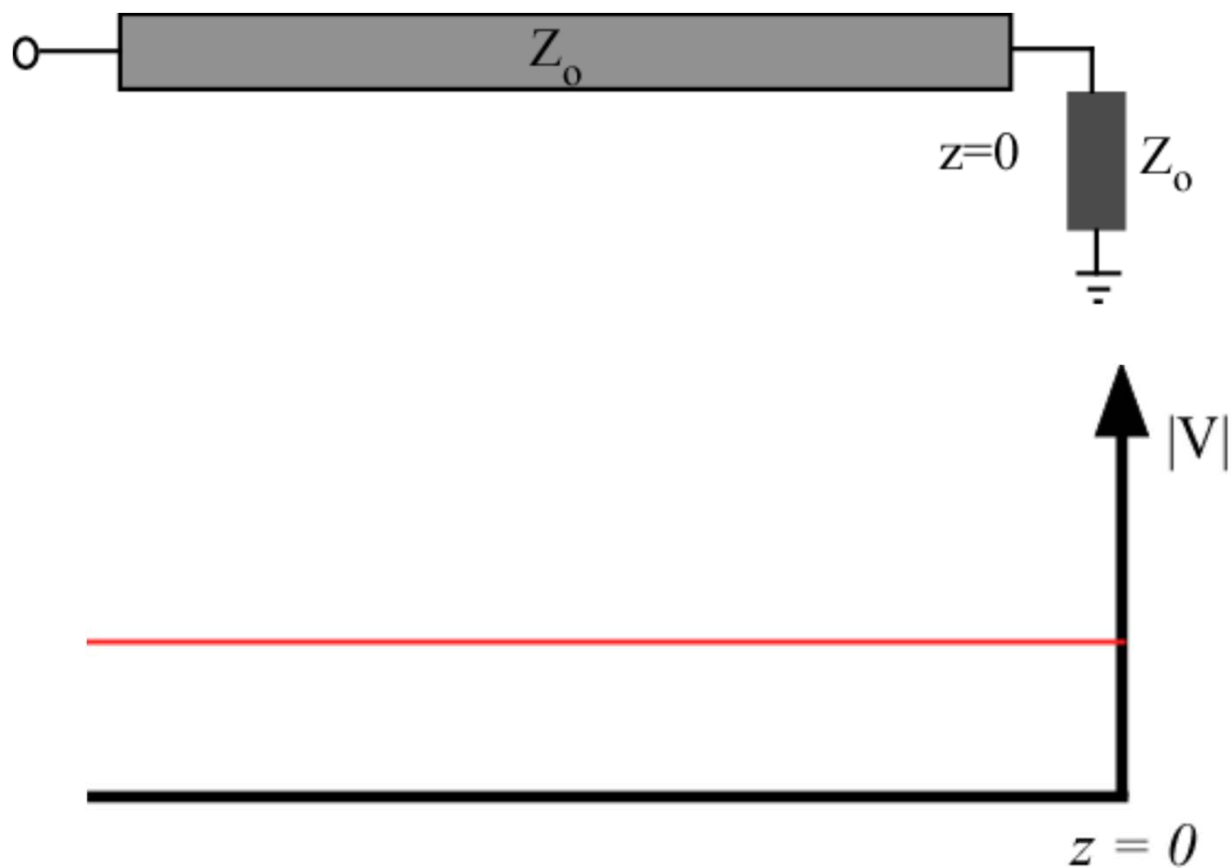
Voltage minimum is reached at load

VSWR – For Open Circuit Load



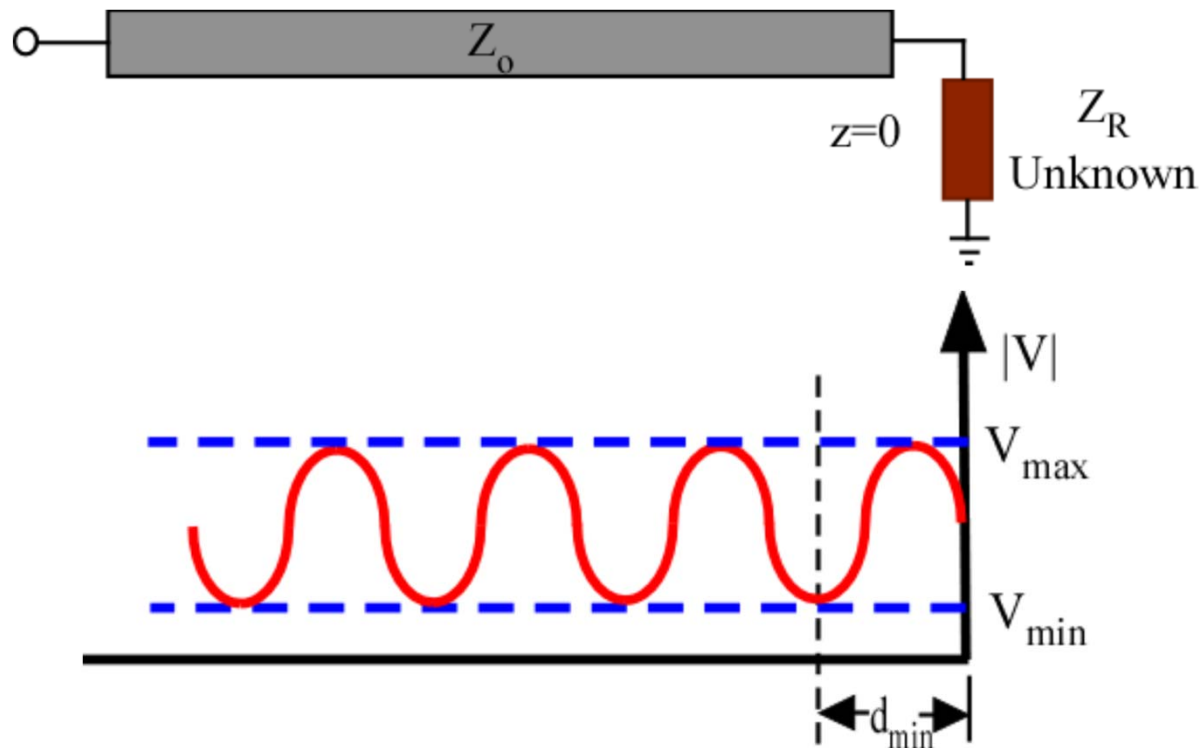
Voltage maximum is reached at load

VSWR – For Open Matched Load



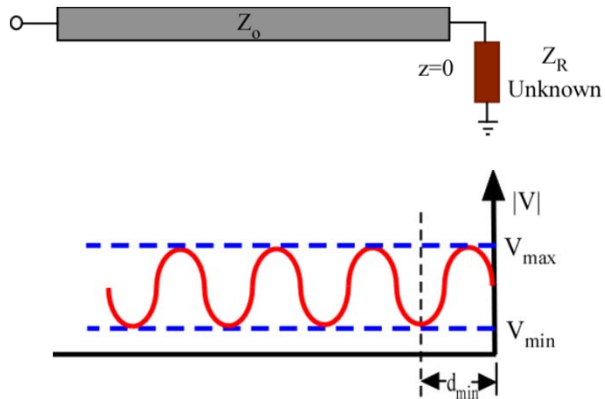
No variation in amplitude along line

Application: Slotted-Line Measurement



- Measure $VSWR = V_{\max}/V_{\min}$
- Measure location of first minimum

Application: Slotted-Line Measurement



At minimum, $\Gamma(z) = \text{pure real} = -|\Gamma_R|$

Therefore, $\Gamma(-d_{\min}) = \Gamma_R e^{-2j\beta d_{\min}} = -|\Gamma_R|$

So, $\Gamma_R = -|\Gamma_R| e^{+2j\beta d_{\min}}$

Since $|\Gamma_R| = \frac{VSWR - 1}{VSWR + 1}$ **then** $\Gamma_R = -\left(\frac{VSWR - 1}{VSWR + 1}\right) e^{+2j\beta d_{\min}}$

and $Z_R = Z_o \left(\frac{1 + \Gamma_R}{1 - \Gamma_R}\right)$

Summary of TL Equations

Voltage

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

Current

$$I(z) = \frac{V_+}{Z_o} e^{-j\beta z} \left[1 - \Gamma_R e^{+2j\beta z} \right]$$

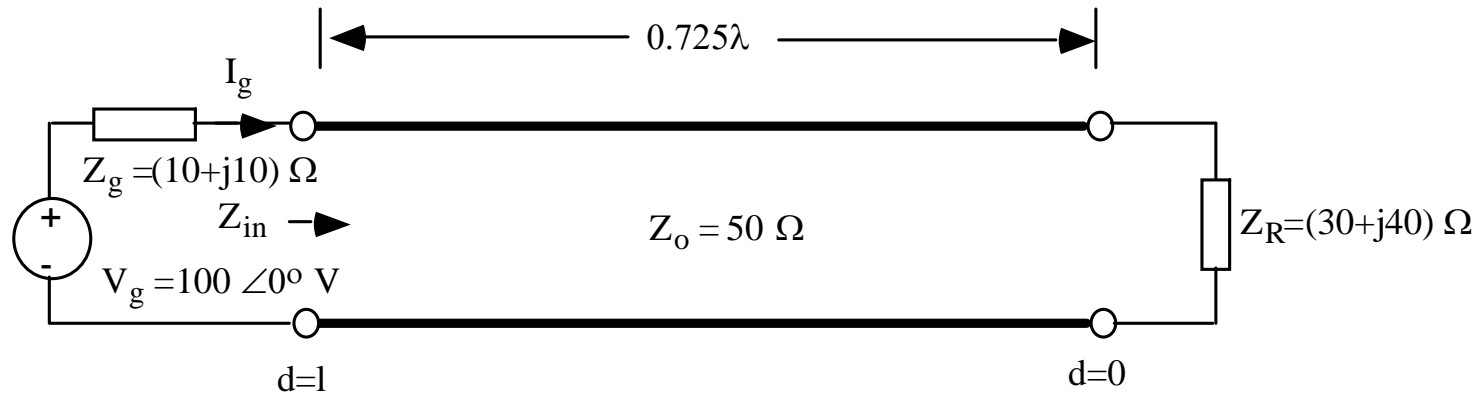
Impedance Transformation → $Z(-l) = Z_o \left[\frac{Z_R + jZ_o \tan \beta l}{Z_o + jZ_R \tan \beta l} \right]$

Reflection Coefficient Transformation → $\Gamma(-l) = \Gamma_R e^{-2j\beta l}$

Reflection Coefficient – to Impedance → $Z(z) = Z_o \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$

Impedance to Reflection Coefficient → $\Gamma(z) = \frac{Z(z) - Z_o}{Z(z) + Z_o}$

Example 1



Consider the transmission line system shown in the figure above.

(a) Find the input impedance Z_{in} .

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o} = \frac{(30 + j40) - 50}{(30 + j40) + 50} = 0.5 \angle 90^\circ$$

$$\Gamma(d = l) = \Gamma_R e^{-2j\beta l} = 0.5 \angle 90^\circ e^{-j\frac{4\pi}{\lambda} \cdot 0.725\lambda} = 0.5 \angle -72^\circ$$

Example 1 – Cont'

$$Z_{in} = Z_o \left[\frac{1 + \Gamma(l)}{1 - \Gamma(l)} \right] = 50 \left[\frac{1 + 0.5 \angle -72^\circ}{1 - 0.5 \angle -72^\circ} \right] \Omega$$

$$Z_{in} = 64.361 \angle -51.738^\circ \Omega = (39.86 - j50.54) \Omega$$

(b) Find the current drawn from the generator

$$I_g = \frac{V_g}{Z_g + Z_{in}} = \frac{100 \angle 0^\circ}{(10 + j10)(39.86 - j50.54)} = 1.552 \angle 39.11^\circ \text{ A}$$

$$I_g = 1.207 + j0.981 \text{ A}$$

Example 1 – Cont'

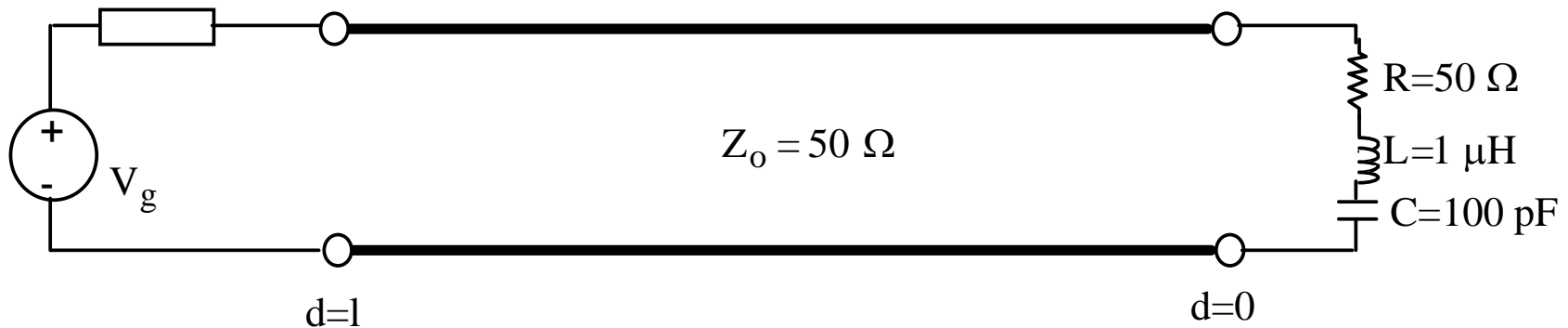
c) Find the time-average power delivered to the load

$$V(l) = Z_{in} I_g = 64.361 \angle 51.73^\circ \times 1.5562 \angle 39.11^\circ = 100.159 \angle -12.62^\circ \frac{1}{2}$$

$$\langle P \rangle = \frac{1}{2} \text{Real} [V(l) I_g^*] = \frac{1}{2} \text{Real} [100.159 \angle -12.62^\circ \times 1.5562 \angle -39.11^\circ]$$

$$\langle P \rangle = \frac{1}{2} \text{Real} [V(l) I_g^*] = 48.26 \text{ W}$$

Example 1 – Cont'



In the system shown above, a line of characteristic impedance 50Ω is terminated by a series R, L, C circuit having the values $R = 50 \Omega$, $L = 1 \mu\text{H}$, and $C = 100 \text{ pF}$.

(d) Find the source frequency f_o for which there are no standing waves on the line.

Example 1 – Cont'

No standing wave on the line if $Z_R = R = Z_o$ which occurs at the resonant frequency of the RLC circuit. Thus,

$$f_o = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10^{-6} \times 10^{-10}}} = \frac{10^8}{2\pi} \text{ Hz} = 15.92 \text{ MHz}$$