ECE 451
Advanced Microwave Measurements

Parallel-Plate Waveguides

Jose E. Schutt-Aine

Electrical & Computer Engineering

University of Illinois

jesa@illinois.edu

ECE·ILLINOIS

Electrical and Computer Engineering
University of Illinois at Urbana-Champaign

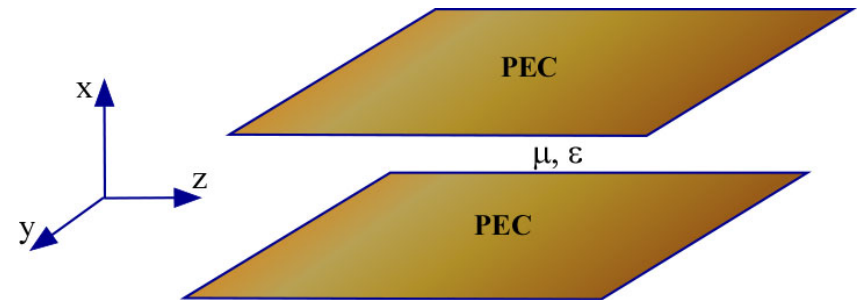# Parallel-Plate Waveguide

Maxwell's Equations ➜ $\nabla^2 \mathbf{E} + \omega^2 \mu\varepsilon\mathbf{E} = 0$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu\varepsilon E_x$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu\varepsilon E_y$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu\varepsilon E_z$$

TE Modes

For a parallel-plate waveguide, the plates are infinite in the y-extent; we need to study the propagation in the z-direction. The following assumptions are made in the wave equation

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

$$\Rightarrow \text{Assume } E_y \text{ only}$$

These two conditions define the **TE modes** and the wave equation is simplified to read

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu \varepsilon E_y \qquad (\yen)$$
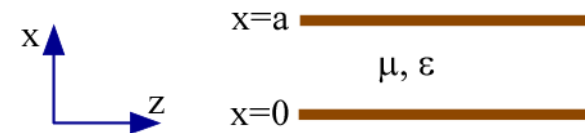
Phasor Solution

General solution (forward traveling wave)

$$E_y(x,z) = e^{-j\beta_z z} \left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right]$$

At $x = 0$, $E_y = 0$ which leads to $A + B = 0$. Therefore, $A = -B = E_o/2j$, where E_o is an arbitrary constant

$$E_y(x,z) = E_o e^{-j\beta_z z} \sin \beta_x x$$

x=a

μ, ε

x=0

a is the distance separating the two PEC plates

Dispersion Relation

At $x = a$, $E_y(x, z) = 0$ ➔ $E_o e^{-j\beta_z z} \sin \beta_x a = 0$

This leads to: $\beta_x a = m\pi$, where $m = 1, 2, 3, ...$

$$\beta_x = \frac{m\pi}{a}$$

Moreover, from the differential equation (¥), we get the *dispersion relation*

$$\beta_z^2 + \beta_x^2 = \omega^2 \mu\varepsilon = \beta^2$$

which leads to $\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$

Guidance Condition

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

where $m = 1, 2, 3 \ldots$ Since propagation is to take place in the z direction, for the wave to propagate, we must have $\beta_z^2 > 0$, or

$$\omega^2 \mu\varepsilon > \left(\frac{m\pi}{a}\right)^2$$

This leads to the following *guidance condition* which will insure wave propagation

$$f > \frac{m}{2a\sqrt{\mu\varepsilon}}$$

Cutoff Frequency

The *cutoff frequency* f_c is defined to be at the onset of propagation

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} \qquad\qquad \lambda_c = \frac{v}{f_c} = \frac{2a}{m}$$

Each mode is referred to as the TE_m mode. It is obvious that there is no TE_0 mode and the first TE mode is the TE_1 mode.

The *cutoff frequency* is the frequency below which the mode associated with the index **m** will not propagate in the waveguide. Different modes will have different cutoff frequencies.

ECE·ILLINOIS
Electrical and Computer Engineering
University of Illinois at Urbana-Champaign

Magnetic Field for TE Modes

From $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

we have $\mathbf{H} = \dfrac{-1}{j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$
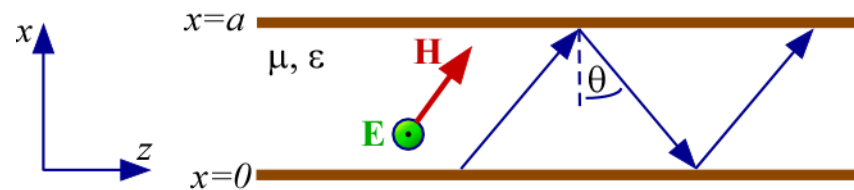
which leads to

$$H_x = -\frac{\beta_z}{\omega\mu} E_o e^{-j\beta_z z} \sin\beta_x x$$

$$H_z = +\frac{j\beta_x}{\omega\mu} E_o e^{-j\beta_z z} \cos\beta_x x$$

The magnetic field for TE modes has 2 components

E & H Fields for TE Modes

As can be seen, there is no H_y component, therefore, the TE solution has E_y, H_x and H_z only.

From the dispersion relation, it can be shown that the propagation vector components satisfy the relations
$\beta_z = \beta \sin\theta$, $\beta_x = \beta \cos\theta$ where θ is the angle of incidence of the propagation vector with the normal to the conductor plates.

Phase and Group Velocities

The phase and group velocities are given by

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{c}{\sqrt{1 - \dfrac{f_c^2}{f^2}}} \qquad \text{and} \qquad v_g = \frac{\partial \omega}{\partial \beta_z} = c\sqrt{1 - \frac{f_c^2}{f^2}}$$

The effective guide impedance is given by:

$$\eta_{TE} = \frac{E_y}{-H_x} = \frac{\eta_o}{\sqrt{1 - \dfrac{f_c^2}{f^2}}}$$
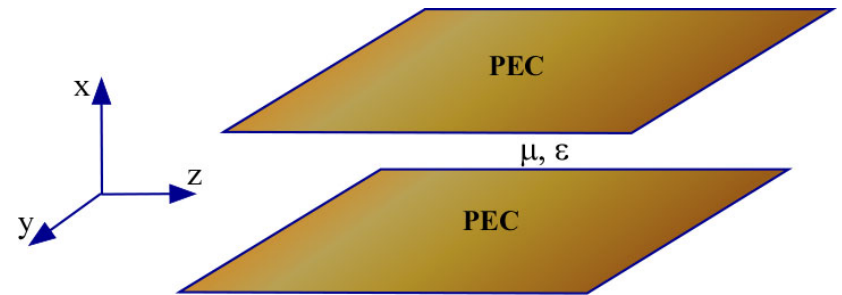
Transverse Magnetic (TM) Modes

The magnetic field also satisfies the wave equation:

Maxwell's Equations ➔ $\nabla^2 \mathbf{H} + \omega^2 \mu \varepsilon \mathbf{H} = 0$

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} = -\omega^2 \mu \varepsilon H_x$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu \varepsilon H_y$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu \varepsilon H_z$$

TM Modes

For TM modes, we assume

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

➔ Assume H_y only

These two conditions define the *TM modes* and the equations are simplified to read

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu \varepsilon H_y$$

General solution (forward traveling wave)

$$H_y(x,z) = e^{-j\beta_z z} \left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right]$$

Electric Field for TM Modes

From $\quad \nabla \times \mathbf{H} = -j\omega\varepsilon\mathbf{E}$

we get $\quad \mathbf{E} = \dfrac{1}{j\omega\varepsilon} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix}$

This leads to

$$E_x(x,z) = \frac{\beta_z}{\omega\varepsilon} e^{-j\beta_z z} \left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right]$$

$$E_z(x,z) = \frac{\beta_x}{\omega\varepsilon} e^{-j\beta_z z} \left[-A e^{-j\beta_x x} + B e^{+j\beta_x x} \right]$$

TM Modes Fields

At $x=0$, $E_z = 0$ which leads to $A = B = H_o/2$ where H_o is an arbitrary constant. This leads to
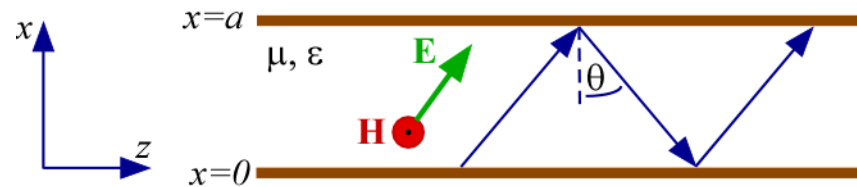
$$H_y(x,z) = H_o e^{-j\beta_z z} \cos \beta_x x$$

$$E_x(x,z) = \frac{\beta_z}{\omega \varepsilon} H_o e^{-j\beta_z z} \cos \beta_x x$$

$$E_z(x,z) = \frac{j\beta_x}{\omega \varepsilon} H_o e^{-j\beta_z z} \sin \beta_x x$$

At $x = a$, $E_z = 0$ which leads to

$$\beta_x a = m\pi, \text{ where } m = 0, 1, 2, 3, \ldots$$

E & H Fields for TM Modes

$$\beta_x = \frac{m\pi}{a}$$

This defines the TM modes which have only H_y, E_x and E_z components.

The effective guide impedance is given by:

$$\eta_{TM} = \frac{E_x}{H_y} = \eta_o \sqrt{1 - \frac{f_c^2}{f^2}}$$

The electric field for TM modes has 2 components

E & H Fields for TM Modes

THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A PARALLEL-PLATE WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

This defines the **TM modes**; each mode is referred to as the TM_m mode. It can be seen from that $m=0$ is a valid choice; it is called the TM_0, or *transverse electromagnetic* or TEM mode. For this mode and,
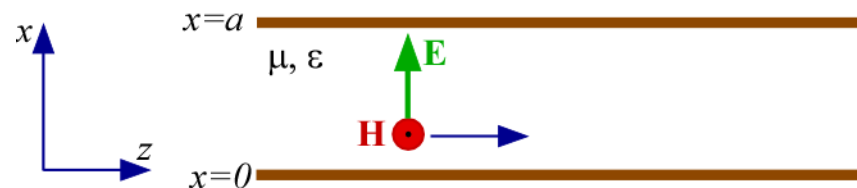
TEM Mode

β_x=0 and $\beta_z = \beta$. There are no x variations of the fields within the waveguide. The TEM mode has a cutoff frequency at DC and is always present in the waveguide.

$$H_y = H_o e^{-j\beta_z z}$$

$$E_x = \frac{\beta_z}{\omega\varepsilon} H_o e^{-j\beta_z z} = \sqrt{\frac{\mu}{\varepsilon}} H_o e^{-j\beta_z z}$$

$$E_z = 0$$

The propagation characteristics of the TEM mode do not vary with frequency

The TEM mode is the *fundamental* mode on a parallel-plate waveguide

Power for TE Modes

Time-Average Poynting Vector $\langle \mathbf{P} \rangle = \frac{1}{2} \operatorname{Re} \{ \mathbf{E} \times \mathbf{H} * \}$

TE modes

$$\langle \mathbf{P} \rangle = \frac{1}{2} \operatorname{Re} \left\{ \hat{\mathbf{y}} E_y \times \left[\hat{\mathbf{x}} H_x^* + \hat{\mathbf{z}} H_z^* \right] \right\}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2} \operatorname{Re} \left\{ \hat{\mathbf{z}} \frac{|E_o|^2}{\omega\mu} \beta_z \sin^2 \beta_x x + \hat{\mathbf{x}} j \frac{|E_o|^2}{\omega\mu} \beta_x \cos \beta_x x \sin \beta_x x \right\}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}} \frac{|E_o|^2}{2\omega\mu} \beta_z \sin^2 \beta_x x$$

Power for TM Modes

TM modes

$$\langle \mathbf{P} \rangle = \frac{1}{2} \mathrm{Re} \left\{ \left[\hat{\mathbf{x}} E_x + \hat{\mathbf{z}} E_z \right] \times \hat{\mathbf{y}} H_y^* \right\}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2} \mathrm{Re} \left\{ \hat{\mathbf{z}} \frac{|H_o|^2}{\omega \varepsilon} \beta_z \cos^2 \beta_x x - \hat{\mathbf{x}} j \frac{|H_o|^2}{\omega \varepsilon} \beta_x \sin \beta_x x \cos \beta_x x \right\}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}} \frac{|H_o|^2}{2\omega \varepsilon} \beta_z \cos^2 \beta_x x$$

The total time-average power is found by integrating <**P**> over the area of interest.