ECE 350

Rectangular Waveguides

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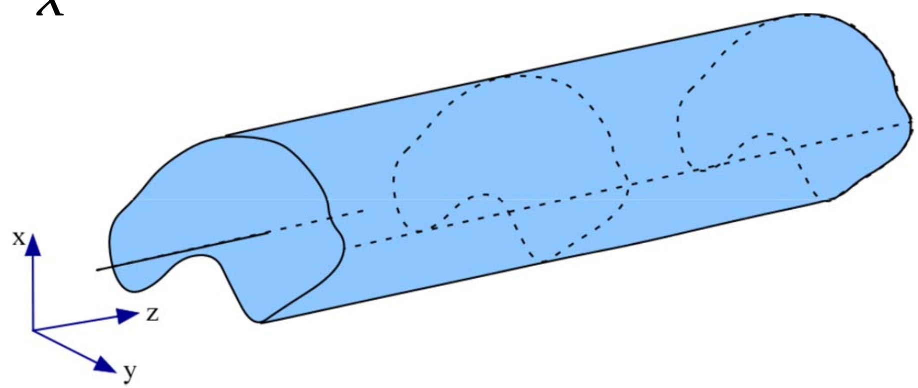
Waveguide

Maxwell's Equations ➔ $\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = \mathbf{0}$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2\mu\varepsilon E_x$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2\mu\varepsilon E_y$$



$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2\mu\varepsilon E_z$$

TE Modes

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set $E_z = 0$. This defines the TE modes.

From $\nabla \times \mathbf{E} = -\mu \dfrac{\partial \mathbf{H}}{\partial t}$, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \Rightarrow +j\beta_z E_y = -j\omega\mu H_x \quad (1)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \Rightarrow -j\beta_z E_x = -j\omega\mu H_y \quad (2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (3)$$

TE Modes

From $\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E}$, we get

$$j\omega\varepsilon \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega\varepsilon E_x \quad (4)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \Rightarrow -j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (6)$$

We want to express all quantities in terms of *H_z*.

TE Modes

From (2), we have $H_y = \frac{\beta_z E_x}{\omega\mu}$

in (4) $\frac{\partial H_z}{\partial y} + j\beta_z^2 \frac{E_x}{\omega\mu} = j\omega\varepsilon E_x$

Solving for E_x $\quad E_x = \frac{j\omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon}\frac{\partial H_z}{\partial y}$

From (1) $\quad H_x = \frac{-\beta_z E_y}{\omega\mu}$

in (5) $j\frac{\beta_z^2 E_y}{\omega\mu} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$

so that $\quad E_y = \frac{-j\omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x}$

TE Modes

$$H_y = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial y}$$

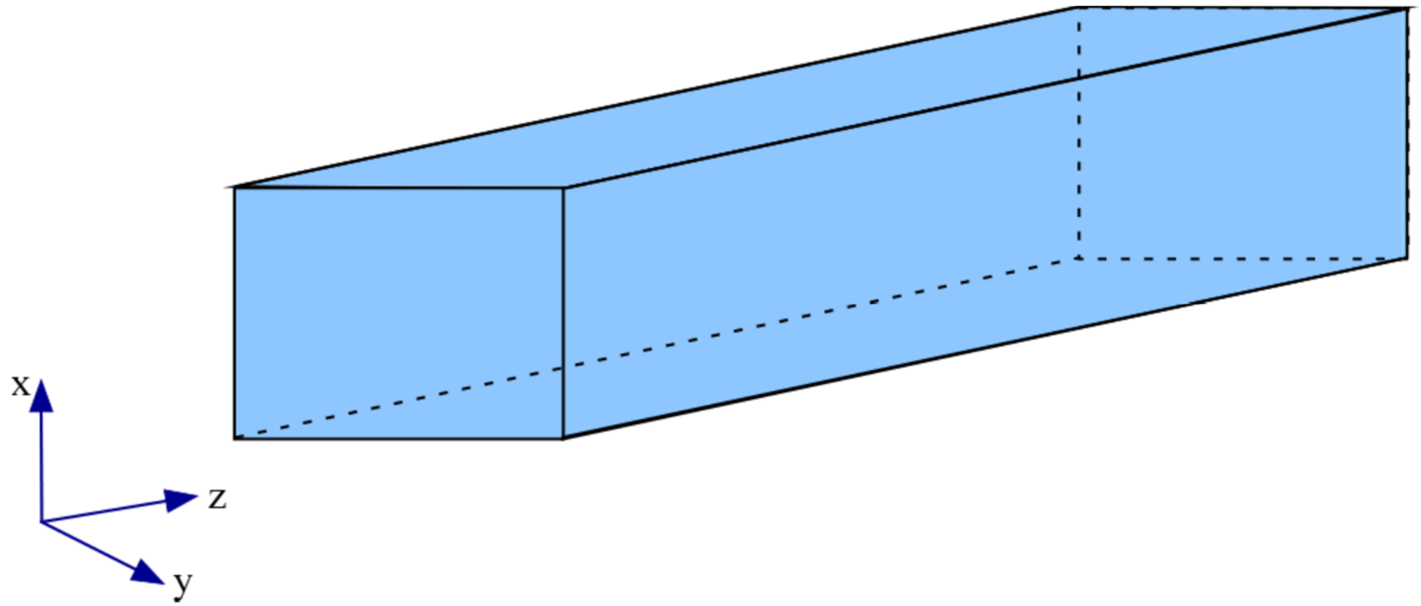
$$H_x = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial x}$$

$$E_z = 0$$

Combining solutions for E_x and E_y into (3) gives

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[\beta_z^2 - \omega^2 \mu\varepsilon\right] H_z \qquad (¥)$$

Rectangular Waveguide



$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[\beta_z^2 - \omega^2 \mu \varepsilon \right] H_z \qquad (¥)$$

If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

TE Modes

The general solution for TE modes with $E_z=0$ is obtained from (¥)

$$H_z = e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y} + De^{+j\beta_y y}\right]$$

$$E_y = \frac{\beta_x \omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon} e^{-j\beta_z z}\left[-Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y} + De^{+j\beta_y y}\right]$$

$$E_x = \frac{-\beta_y \omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon} e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[-Ce^{-j\beta_y y} + De^{+j\beta_y y}\right]$$

At y=0, Ex=0 which leads to $C=D$

At x=0, Ey=0 which leads to $A=B$

TE Modes

$$H_z = H_o e^{-j\beta_z z} \cos\beta_x x \cos\beta_y y \qquad (\S)$$

$$E_y = \frac{j\beta_x \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} H_o e^{-j\beta_z z} \sin\beta_x x \cos\beta_y y$$

$$E_x = \frac{-j\beta_y \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} H_o e^{-j\beta_z z} \cos\beta_x x \sin\beta_y y$$

At x=a, E_y=0 which leads to $\beta_x = \dfrac{m\pi}{a}$

At y=b, E_x=0 which leads to $\beta_y = \dfrac{n\pi}{b}$

The general solution for TE modes with $E_z=0$ is

Dispersion Relation

The dispersion relation is obtained by placing (§) in (¥)

$$\beta_z^2 + \beta_x^2 + \beta_z^2 = \omega^2 \mu\varepsilon \qquad (23)$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \omega^2 \mu\varepsilon \quad (24)$$

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad (25)$$

The guidance condition is

$$\omega^2 \mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad (26)$$

Guidance Condition

or $f > f_c$ where f_c is the cutoff frequency of the TE_{mn} mode given by the relation

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The TE_{mn} mode will not propagate unless f is greater than f_c.

Obviously, different modes will have different cutoff frequencies.

TM Mode

The transverse magnetic modes for a general waveguide are obtained by assuming $H_z = 0$. By duality with the TE modes, we have

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \left[\beta_z^2 - \omega^2 \mu \varepsilon \right] E_z$$

$$E_z = e^{-j\beta_z z} \left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[C e^{-j\beta_y y} + D e^{+j\beta_y y} \right]$$

TM Mode

The boundary conditions are

At $x=0$, $Ez=0$ which leads to $A=-B$

At $y=0$, $Ez=0$ which leads to $C=-D$

At $x=a$, $Ez=0$ which leads to $\beta_x = \dfrac{m\pi}{a}$

At $y=b$, $Ez=0$ which leads to $\beta_y = \dfrac{n\pi}{b}$

TM and TE Modes

so that the generating equation for the TM_{mn} modes is

$$E_z = E_o e^{-j\beta_z z} \sin \beta_x x \sin \beta_y y$$

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

For additional information on the field equations see **Rao (6th Edition), page 607, Table 9.1**.

TE and TM Modes

There is no TE_{00} mode

There are no TM_{m0} or TM_{0n} modes

The first TE mode is the TE_{10} mode

The first TM mode is the TM_{11} mode

Impedance of a Waveguide

For a TE mode, we define the transverse impedance as

$$\eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta_z}$$

From the relationship for β_z and using

we get $$f_c^2 = \frac{1}{4\mu\varepsilon}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$$

$$\eta_{gTE} = \frac{\eta}{\sqrt{1-\frac{f_c^2}{f^2}}}$$ where η is th intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

Impedance of a Waveguide

Analogously, for TM modes, it can be shown that

$$\eta_{gTM} = \eta\sqrt{1-\frac{f_c^2}{f^2}}$$

Power Flow in a Waveguide

TE_{10} Mode

The time-average Poynting vector for the TE_{10} mode in a rectangular waveguide is given by

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right] = \hat{\mathbf{z}}\frac{|E_o|^2}{2}\frac{\beta_z}{\omega\mu}\sin^2\frac{\pi x}{a}$$

$$\langle Power \rangle = \int_0^a\int_0^b \frac{|E_o|^2}{2}\frac{\beta_z}{\omega\mu}\sin^2\frac{\pi x}{a}dxdy$$

$$\langle Power \rangle = \frac{|E_o|^2}{4}\frac{\beta_z ab}{\omega\mu} = \frac{|E_o|^2}{4}\frac{ab}{\eta_{gTE_{10}}}$$

The time-average power flow in a waveguide is proportional to its cross-section area.