ECE 453
Wireless Communication Systems

Angle Modulation and Bessel Functions

Jose E. Schutt-Aine
Electrical & Computer Engineering
University of Illinois
jesa@illinois.edu
Phase Modulation (PM)

In phase modulation, the instantaneous phase deviation of the modulated signal from its unmodulated value is proportional to the instantaneous amplitude of the modulating signal.

\[ F(t) = A(t) \cos(\omega_c t + \Theta(t)) = A(t) \cos(\phi(t)) \]

**Phase**

\[ \Theta(t) = k_{\Theta} v_m(t) = k_{\Theta} V_m v(t) \]

\[ F_{PM}(t) = A \cos(\omega_c t + m_p v(t)) \]

\[ \Theta(t) = m_p v(t) \]

**Instantaneous frequency**

\[ \omega(t) = \omega_c + \frac{d\Theta(t)}{dt} = \omega_c + m_p \frac{dv(t)}{dt} \]
Phase Modulation

Assume message is sinusoidal
Frequency Modulation (FM)

Frequency modulation results when the deviation $\delta \omega$ of the instantaneous frequency $\omega(t)$ from the carrier frequency $\omega_c$ is directly proportional to the instantaneous amplitude of the modulating voltage.

$$F(t) = A(t) \cos[\omega_c t + \Theta(t)] = A(t) \cos \phi(t)$$

**Instantaneous frequency**

$$\omega(t) = \frac{d\phi}{dt} = \omega_c + \frac{d\Theta(t)}{dt}$$

$$\delta \omega(t) = \omega(t) - \omega_c = \frac{d\Theta(t)}{dt} = k_\omega v_m(t)$$

**Phase**

$$\Theta(t) = \int_0^t k_\omega v_m dt + \Theta(0)$$

$$F_{FM}(t) = A \cos \left[ \omega_c t + k_\omega \int_0^t v_m(t) dt \right]$$
Frequency Modulation

**Message**
Assume message is sinusoidal

**Frequency**

**Angle**

Assume message is sinusoidal.
Modulation Index for FM

Signal for FM can be put in the form

\[ F_{FM}(t) = A \cos \left[ \omega_c t + \frac{\Delta f}{f_m} \sin \omega_m t \right] \]

The modulation index is defined as

\[ m_f = \frac{\Delta f}{f_m} \]

Assume message is sinusoidal

It is the maximum phase deviation

So, the expression becomes

\[ F_{FM}(t) = A \cos(\omega_c t + m_f \sin \omega_m t) \]
Angle Modulation

In general for an angle-modulated signal, we have

\[ F_\Theta(t) = A \cos(\omega_c t + m_\Theta \sin \omega_m t) \]

This can be expanded to take the form

\[ F_\Theta(t) = V_c \{ J_0(m_\Theta) \cos \omega_c t \]
\[ + J_1(m_\Theta) [\cos(\omega_c + \omega_m) t - \cos(\omega_c - \omega_m) t] \]
\[ + J_2(m_\Theta) [\cos(\omega_c + 2\omega_m) t - \cos(\omega_c - 2\omega_m) t] \]
\[ + J_3(m_\Theta) [\cos(\omega_c + 3\omega_m) t - \cos(\omega_c - 3\omega_m) t] \]
\[ + J_4(m_\Theta) [\cos(\omega_c + 4\omega_m) t - \cos(\omega_c - 4\omega_m) t] \]
\[ + \ldots \} \]

The \( J_n(m_\Theta) \) are Bessel functions of the first kind
Spectrum of Angle Modulated Signal

Signal for PM or FM takes the form

\[ F_\Theta(t) = V_c \cos[\omega_c t + m_\Theta \sin \omega_m t] \]

From trigonometric identity,

\[ F_\Theta(t) = V_c \left[ \cos \omega_c t \cos(m_\Theta \sin \omega_m t) - \sin \omega_c t \cos(m_\Theta \sin \omega_m t) \right] \]

Consider the function

\[ g(t) = e^{-jm_\Theta \sin \omega_m t} \]

Its exponential Fourier series is represented by

\[ g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_nt} \]
Spectrum of Angle Modulated Signal

Where the coefficients \( c_n \) are given by

\[
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} g(t) e^{-j\omega t} d(\omega t)
\]

The Bessel function of the first kind is defined by

\[
J_n(m_\Theta) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{jm_\Theta \sin \omega_m t} \times e^{-j\omega t} d(\omega t)
\]

Thus for \( g(t) \), \( c_n \) is given by

\[
c_n = J_n(m_\Theta)
\]

And \( g(t) \) can be expressed by

\[
g(t) = \sum_{n=-\infty}^{\infty} J_n(m_\Theta) e^{j\omega t}
\]
Spectrum of Angle Modulated Signal

If the $+n$ and $-n$ components of $g(t)$ are summed, it can be shown that

$$J_{-n}(m_\Theta) = (-1)^n J_n(m_\Theta)$$

Consequently for $n$ even

$$J_{-n}(m_\Theta)e^{-jn\omega_m t} + J_n(m_\Theta)e^{jn\omega_m t}$$

$$= J_n(m_\Theta)\left[ e^{jn\omega_m t} + e^{-jn\omega_m t} \right] = 2J_n(m_\Theta)\cos n\omega_m t$$

For $n$ odd, the summation yields

$$J_{-n}(m_\Theta)e^{-jn\omega_m t} + J_n(m_\Theta)e^{jn\omega_m t}$$

$$= J_n(m_\Theta)\left[ e^{jn\omega_m t} - e^{-jn\omega_m t} \right] = j2J_n(m_\Theta)\sin n\omega_m t$$
Spectrum of Angle Modulated Signal

Therefore $g(t)$ can be expressed as

$$g(t) = J_0 (m_\Theta) + 2 \sum_{n \text{ even}} J_n (m_\Theta) \cos n\omega_m t$$

$$+ 2j \sum_{n \text{ odd}} J_n (m_\Theta) \sin n\omega_m t$$

By application of Euler’s theorem

$$g(t) = \cos(m_\Theta \sin \omega_m t) + j \sin(m_\Theta \sin \omega_m t)$$

$$\cos(m_\Theta \sin n\omega_m t) = J_0 (m_\Theta) + 2 \sum_{n \text{ even}} J_n (m_\Theta) \cos n\omega_m t$$

$$\sin(m_\Theta \sin n\omega_m t) = 2 \sum_{n \text{ odd}} J_n (m_\Theta) \sin n\omega_m t$$
Spectrum of Angle Modulated Signal

After substitution into $F_\Theta(t)$

$$F_\Theta(t) = A \cos(\omega_c t + m_\Theta \sin \omega_m t)$$

this yields

$$F_\Theta(t) = V_c \{J_0 (m_\Theta) \cos \omega_c t$$

$$+ J_1 (m_\Theta) \left[ \cos(\omega_c + \omega_m) t - \cos(\omega_c - \omega_m) t \right]$$

$$+ J_2 (m_\Theta) \left[ \cos(\omega_c + 2\omega_m) t - \cos(\omega_c - 2\omega_m) t \right]$$

$$+ J_3 (m_\Theta) \left[ \cos(\omega_c + 3\omega_m) t - \cos(\omega_c - 3\omega_m) t \right]$$

$$+ J_4 (m_\Theta) \left[ \cos(\omega_c + 4\omega_m) t - \cos(\omega_c - 4\omega_m) t \right]$$

$$+ \ldots \}$$
Bessel Functions of the First Kind

\[ J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r! \Gamma(n+r+1)} \]

\[ \Gamma(n+1) = n! \]
Zeros of Bessel Functions

<table>
<thead>
<tr>
<th>Order $n^{\text{th}}$ root</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.405</td>
<td>3.832</td>
<td>5.136</td>
</tr>
<tr>
<td>2</td>
<td>5.520</td>
<td>7.016</td>
<td>8.417</td>
</tr>
<tr>
<td>3</td>
<td>8.654</td>
<td>13.323</td>
<td>11.620</td>
</tr>
</tbody>
</table>
Selecting Modulation index

- Carrier may vanish as modulation index varies
- $\Delta f$ and $f_m$ can be chosen so as to make $J_0(m_\varphi)=0$
- Modulator can be adjusted until carrier vanishes

\[
F_\varphi(t) = V_c \{ J_0(m_\varphi) \cos \omega_c t 
+ J_1(m_\varphi) \left[ \cos(\omega_c + \omega_m)t - \cos(\omega_c - \omega_m)t \right] 
+ J_2(m_\varphi) \left[ \cos(\omega_c + 2\omega_m)t - \cos(\omega_c - 2\omega_m)t \right] 
+ J_3(m_\varphi) \left[ \cos(\omega_c + 3\omega_m)t - \cos(\omega_c - 3\omega_m)t \right] 
+ J_4(m_\varphi) \left[ \cos(\omega_c + 4\omega_m)t - \cos(\omega_c - 4\omega_m)t \right] 
+ \ldots \} 
\]
Bandwidth of Angle Modulated Signal

Bandwidth can be approximated by

\[ BW \approx 2\left(m_\Theta + 1\right)f_m \approx 2\left(\Delta f_{\text{max}} + f_m\right) \]
Spectrum of Frequency Modulated Carrier

$\beta$ is modulation index