ECE 453
Wireless Communication Systems

Angle Modulation and Bessel Functions

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Phase Modulation (PM)

In phase modulation, the instantaneous phase deviation of the modulated signal from its unmodulated value is proportional to the instantaneous amplitude of the modulating signal.

$$F(t) = A(t)\cos\left[\omega_c t + \Theta(t)\right] = A(t)\cos\phi(t)$$

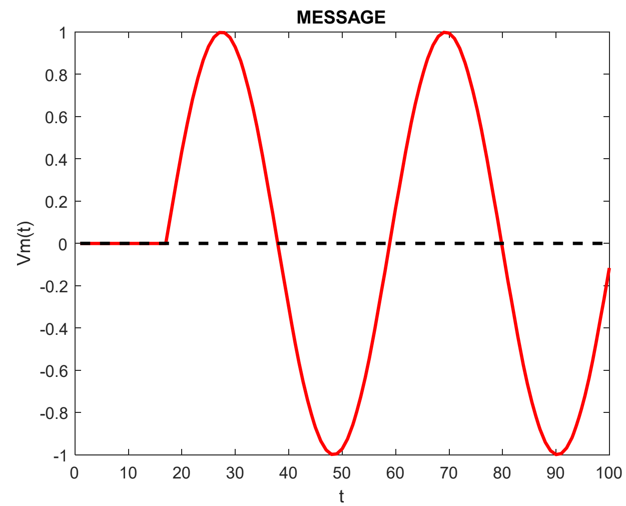
Phase ➡ $\Theta(t) = k_{\Theta} v_m(t) = k_{\Theta} V_m v(t)$

$$F_{PM}(t) = A\cos\left[\omega_c t + m_p v(t)\right]$$

$$\Theta(t) = m_p v(t)$$

Instantaneous frequency ➡ $\omega(t) = \omega_c + \dfrac{d\Theta(t)}{dt} = \omega_c + m_p \dfrac{dv(t)}{dt}$

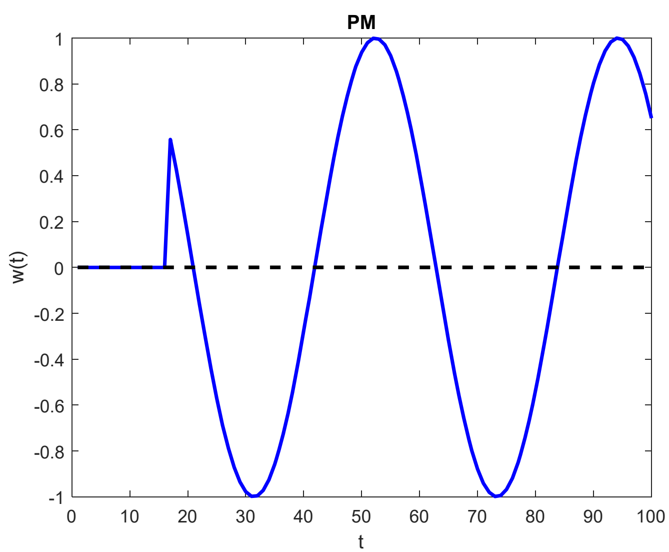
Phase Modulation



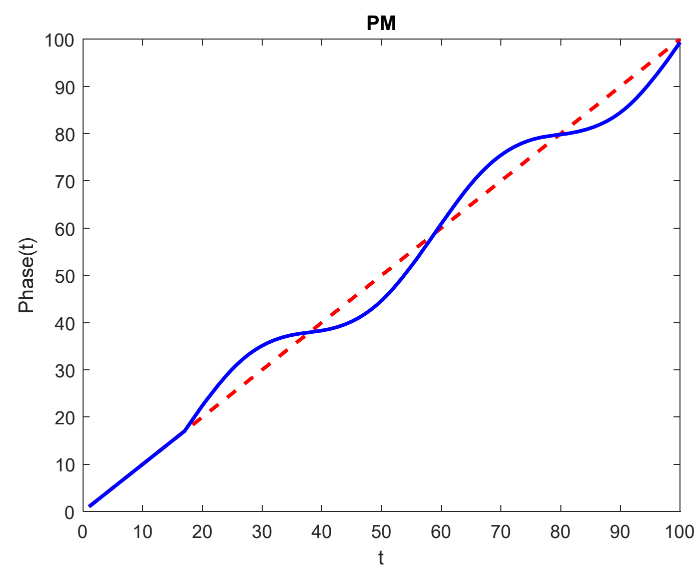
Message

Assume message is sinusoidal

Frequency



Angle



Frequency Modulation (FM)

Frequency modulation results when the deviation $\delta\omega$ of the instantaneous frequency $\omega(t)$ from the carrier frequency ω_c is directly proportional to the instantaneous amplitude of the modulating voltage.

$$F(t) = A(t)\cos\left[\omega_c t + \Theta(t)\right] = A(t)\cos\phi(t)$$

Instantaneous frequency ➡ $$\omega(t) = \frac{d\phi}{dt} = \omega_c + \frac{d\Theta(t)}{dt}$$

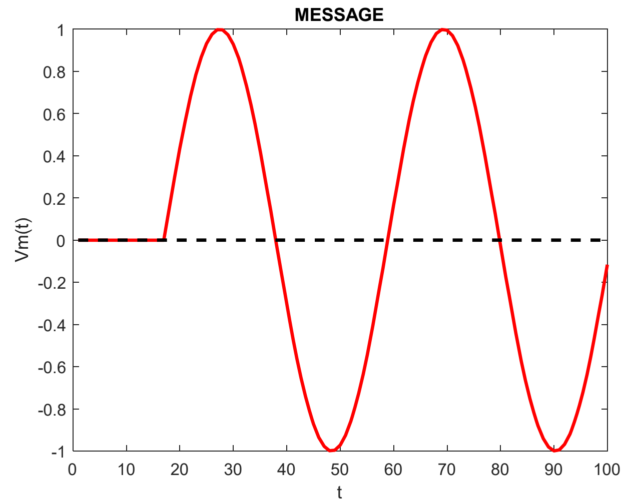
$$\delta\omega(t) = \omega(t) - \omega_c = \frac{d\Theta(t)}{dt} = k_\omega v_m(t)$$

Phase ➡ $$\Theta(t) = \int_0^t k_\omega v_m dt + \Theta(0)$$

$$F_{FM}(t) = A\cos\left[\omega_c t + k_\omega \int_0^t v_m(t)dt\right]$$

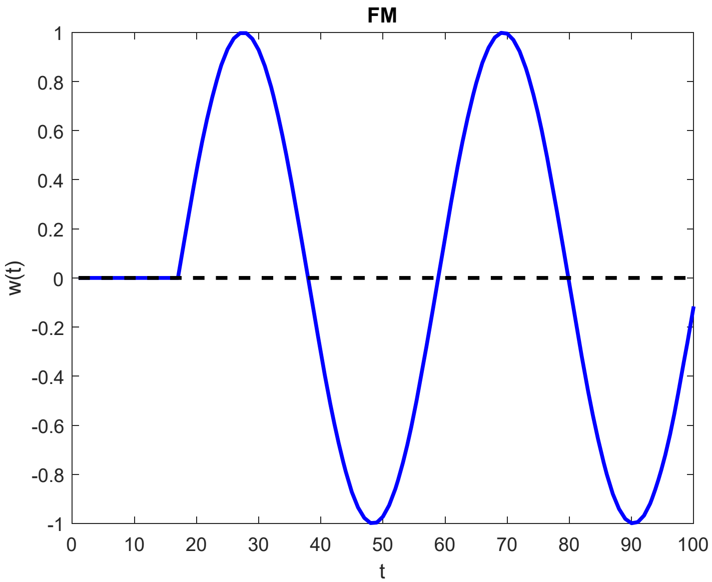
Frequency Modulation

Message

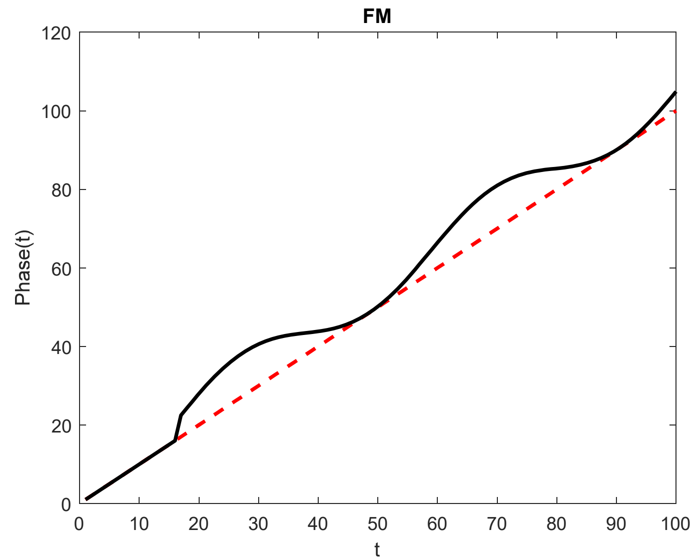


Assume message is sinusoidal

Frequency



Angle



Modulation Index for FM

Signal for FM can be put in the form

$$F_{FM}(t) = A\cos\left[\omega_c t + \frac{\Delta f}{f_m}\sin\omega_m t\right]$$

Assume message is sinusoidal

The modulation index is defined as

$$m_f = \frac{\Delta f}{f_m}$$

It is the maximum phase deviation

So, the expression becomes

$$F_{FM}(t) = A\cos(\omega_c t + m_f \sin\omega_m t)$$

Angle Modulation

In general for an angle-modulated signal, we have

$$F_{\Theta}(t) = A\cos(\omega_c t + m_{\Theta}\sin\omega_m t)$$

This can be expanded to take the form

$$\begin{aligned} F_{\Theta}(t) = V_c\{ & J_o(m_{\Theta})\cos\omega_c t \\ & + J_1(m_{\Theta})\left[\cos(\omega_c + \omega_m)t - \cos(\omega_c - \omega_m)t\right] \\ & + J_2(m_{\Theta})\left[\cos(\omega_c + 2\omega_m)t - \cos(\omega_c - 2\omega_m)t\right] \\ & + J_3(m_{\Theta})\left[\cos(\omega_c + 3\omega_m)t - \cos(\omega_c - 3\omega_m)t\right] \\ & + J_4(m_{\Theta})\left[\cos(\omega_c + 4\omega_m)t - \cos(\omega_c - 4\omega_m)t\right] \\ & + \ldots\} \end{aligned}$$

The $J_n(m_{\Theta})$ are Bessel functions of the first kind

Spectrum of Angle Modulated Signal

Signal for PM or FM takes the form

$$F_{\Theta}(t) = V_c \cos\left[\omega_c t + m_{\Theta} \sin \omega_m t\right]$$

From trigonometric identity,

$$F_{\Theta}(t) = V_c \left[\cos \omega_c t \cos\left(m_{\Theta} \sin \omega_m t\right) - \sin \omega_c t \sin\left(m_{\Theta} \sin \omega_m t\right)\right]$$

Consider the function

$$g(t) = e^{-jm_{\Theta} \sin \omega_m t}$$

Its exponential Fourier series is represented by

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

Spectrum of Angle Modulated Signal

Where the coefficients c_n are given by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-jn\omega t}\,d(\omega t)$$

The Bessel function of the first kind is defined by

$$J_n(m_\Theta) = \frac{1}{2\pi}\int_0^{2\pi} e^{jm_\Theta \sin\omega_m t} \times e^{-jn\omega t}\,d(\omega t)$$

Thus for $g(t)$, c_n is given by

$$c_n = J_n(m_\Theta)$$

And $g(t)$ can be expressed by

$$g(t) = \sum_{n=-\infty}^{\infty} J_n(m_\Theta)e^{jn\omega t}$$

Spectrum of Angle Modulated Signal

If the +*n* and –*n* components of *g(t)* are summed, it can be shown that

$$J_{-n}(m_\Theta) = (-1)^n J_n(m_\Theta)$$

Consequently for *n even*

$$J_{-n}(m_\Theta)e^{-jn\omega_m t} + J_n(m_\Theta)e^{+jn\omega_m t}$$
$$= J_n(m_\Theta)\left[e^{+jn\omega_m t} + e^{-jn\omega_m t}\right] = 2J_n(m_\Theta)\cos n\omega_m t$$

For *n odd,* the summation yields

$$-J_{-n}(m_\Theta)e^{-jn\omega_m t} + J_n(m_\Theta)e^{+jn\omega_m t}$$
$$= J_n(m_\Theta)\left[e^{+jn\omega_m t} - e^{-jn\omega_m t}\right] = j2J_n(m_\Theta)\sin n\omega_m t$$

Spectrum of Angle Modulated Signal

Therefore *g(t)* can be expressed as

$$g(t) = J_o(m_\Theta) + 2\sum_{n\,even} J_n(m_\Theta)\cos n\omega_m t$$

$$+2j\sum_{n\,odd} J_n(m_\Theta)\sin n\omega_m t$$

By application of Euler's theorem

$$g(t) = \cos(m_\Theta \sin\omega_m t) + j\sin(m_\Theta \sin\omega_m t)$$

$$\cos(m_\Theta \sin\omega_m t) = J_o(m_\Theta) + 2\sum_{n\,even} J_n(m_\Theta)\cos n\omega_m t$$

$$\sin(m_\Theta \sin\omega_m t) = 2\sum_{n\,odd} J_n(m_\Theta)\sin n\omega_m t$$

Spectrum of Angle Modulated Signal

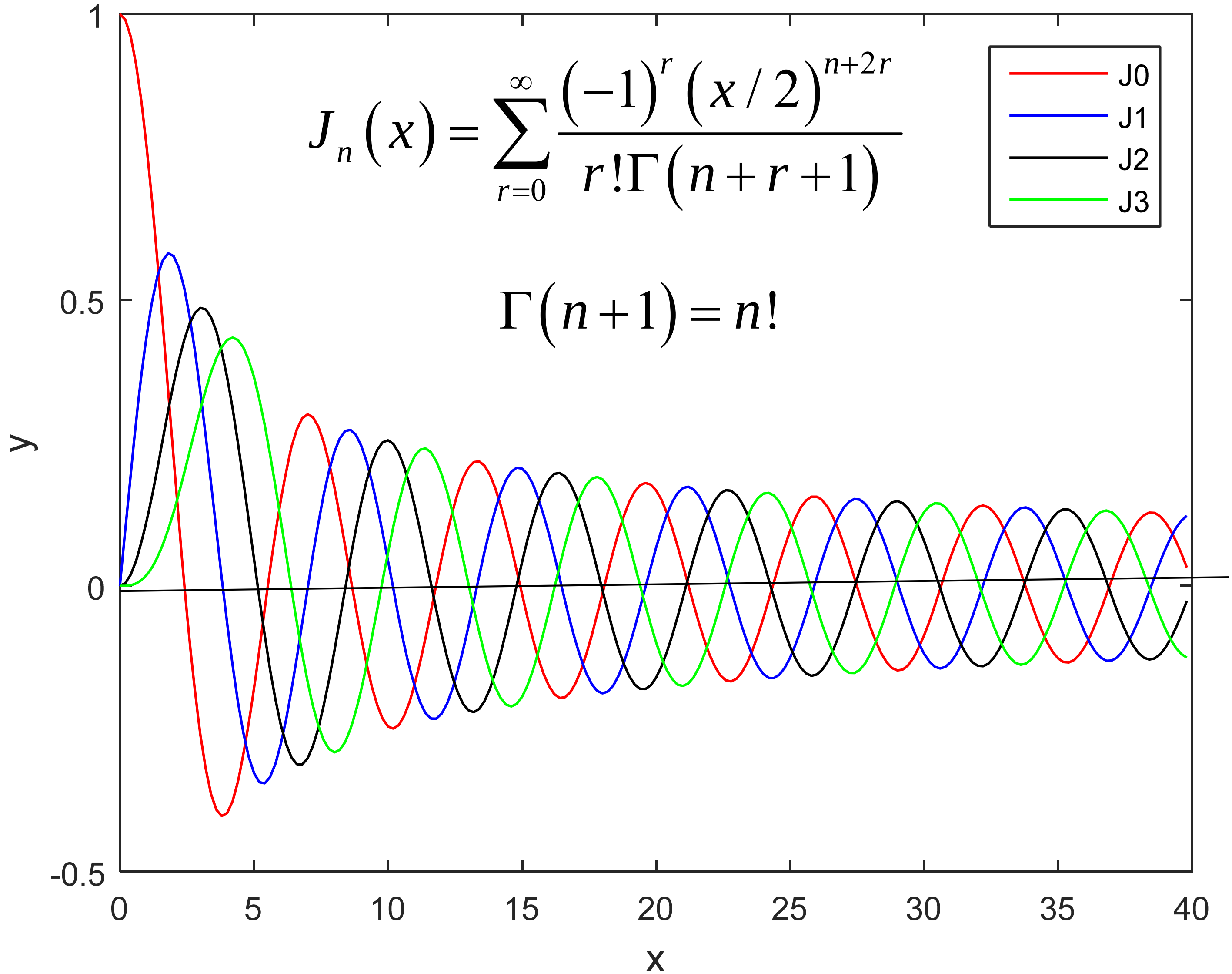
After substitution into $F_\Theta(t)$

$$F_\Theta(t) = A\cos(\omega_c t + m_\Theta \sin\omega_m t)$$

this yields

$$\begin{aligned} F_\Theta(t) = V_c\{ & J_o(m_\Theta)\cos\omega_c t \\ & + J_1(m_\Theta)\left[\cos(\omega_c + \omega_m)t - \cos(\omega_c - \omega_m)t\right] \\ & + J_2(m_\Theta)\left[\cos(\omega_c + 2\omega_m)t - \cos(\omega_c - 2\omega_m)t\right] \\ & + J_3(m_\Theta)\left[\cos(\omega_c + 3\omega_m)t - \cos(\omega_c - 3\omega_m)t\right] \\ & + J_4(m_\Theta)\left[\cos(\omega_c + 4\omega_m)t - \cos(\omega_c - 4\omega_m)t\right] \\ & + \ldots\} \end{aligned}$$

Bessel Functions of the First Kind



$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\Gamma(n+r+1)}$$

$$\Gamma(n+1) = n!$$

Zeros of Bessel Functions

Order n →
l^{th} **root** ↓

	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	13.323	11.620

Spectrum of Angle Modulated Signal

- **Selecting Modulation index**
  - Carrier may vanish as modulation index varies
  - Δf and f_m can be chosen so as to make $J_0(m_\Theta)=0$
  - Modulator can be adjusted until carrier vanishes

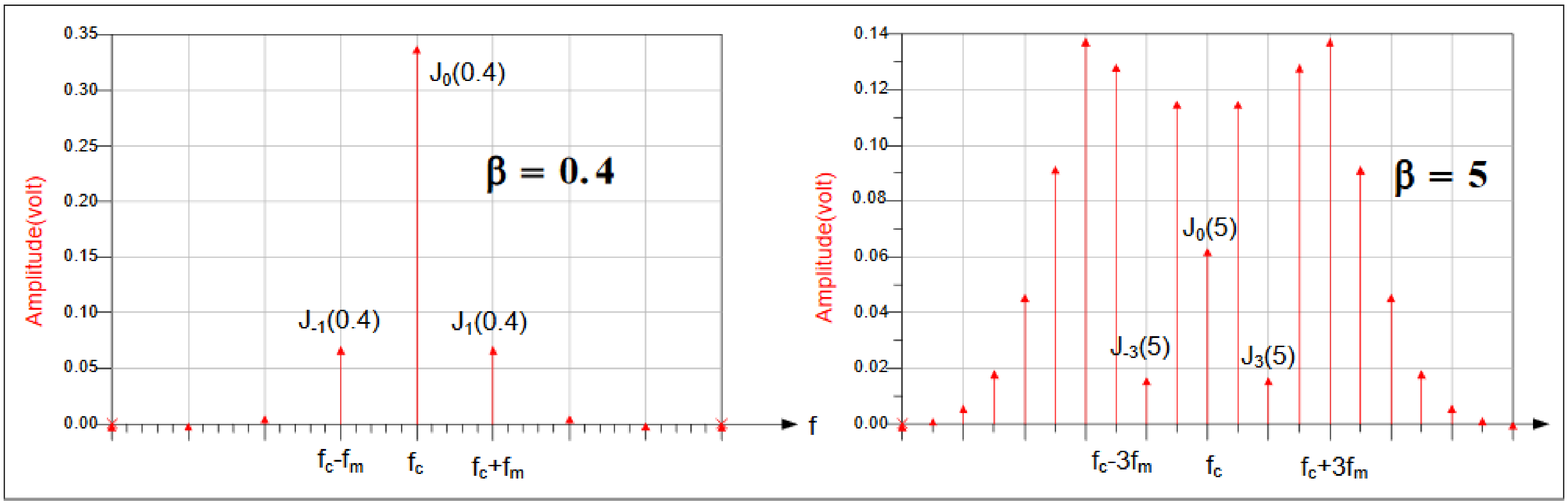
$$
\begin{aligned}
F_\Theta(t) = V_c\{&J_o(m_\Theta)\cos\omega_c t \\
&+ J_1(m_\Theta)\left[\cos(\omega_c+\omega_m)t - \cos(\omega_c-\omega_m)t\right] \\
&+ J_2(m_\Theta)\left[\cos(\omega_c+2\omega_m)t - \cos(\omega_c-2\omega_m)t\right] \\
&+ J_3(m_\Theta)\left[\cos(\omega_c+3\omega_m)t - \cos(\omega_c-3\omega_m)t\right] \\
&+ J_4(m_\Theta)\left[\cos(\omega_c+4\omega_m)t - \cos(\omega_c-4\omega_m)t\right] \\
&+ \ldots\}
\end{aligned}
$$

Bandwidth of Angle Modulated Signal

Bandwidth can be approximated by

$$BW \simeq 2\left(m_{\Theta}+1\right) f_m \simeq 2\left(\Delta f_{\max} + f_m\right)$$

Spectrum of Frequency Modulated Carrier



β is modulation index