ECE 453

Wireless Communication Systems

Angle Modulation and Bessel Functions

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Phase Modulation (PM)

In phase modulation, the instantaneous phase deviation of the modulated signal from its unmodulated value is proportional to the instantaneous amplitude of the modulating signal.

\[
F(t) = A(t) \cos(\omega_c t + \Theta(t)) = A(t) \cos \phi(t)
\]

**Phase**  \[\Theta(t) = k_\Theta v_m(t) = k_\Theta V_m v(t)\]

**Instantaneous frequency**  \[\omega(t) = \omega_c + \frac{d\Theta(t)}{dt} = \omega_c + m_p \frac{dv(t)}{dt}\]
Phase Modulation

Assume message is sinusoidal

Message

Frequency

Angle
Frequency Modulation (FM)

Frequency modulation results when the deviation $\delta \omega$ of the instantaneous frequency $\omega(t)$ from the carrier frequency $\omega_c$ is directly proportional to the instantaneous amplitude of the modulating voltage.

$$F(t) = A(t) \cos[\omega_c t + \Theta(t)] = A(t) \cos \phi(t)$$

**Instantaneous frequency**

$$\omega(t) = \frac{d\phi}{dt} = \omega_c + \frac{d\Theta(t)}{dt}$$

$$\delta \omega(t) = \omega(t) - \omega_c = \frac{d\Theta(t)}{dt} = k_\omega v_m(t)$$

**Phase**

$$\Theta(t) = \int_0^t k_\omega v_m dt + \Theta(0)$$

$$F_{FM}(t) = A \cos \left[ \omega_c t + k_\omega \int_0^t v_m(t) dt \right]$$
Assume message is sinusoidal.
Modulation Index for FM

Signal for FM can be put in the form

\[ F_{FM}(t) = A \cos \left( \omega_c t + \frac{\Delta f}{f_m} \sin \omega_m t \right) \]

The modulation index is defined as

\[ m_f = \frac{\Delta f}{f_m} \]

So, the expression becomes

\[ F_{FM}(t) = A \cos(\omega_c t + m_f \sin \omega_m t) \]
Angle Modulation

In general for an angle-modulated signal, we have

$$F_\Theta(t) = A \cos(\omega_c t + m_\Theta \sin \omega_m t)$$

This can be expanded to take the form

$$F_\Theta(t) = V_c \{ J_0(m_\Theta) \cos \omega_c t \\
+ J_1(m_\Theta) [\cos(\omega_c + \omega_m) t - \cos(\omega_c - \omega_m) t] \\
+ J_2(m_\Theta) [\cos(\omega_c + 2\omega_m) t - \cos(\omega_c - 2\omega_m) t] \\
+ J_3(m_\Theta) [\cos(\omega_c + 3\omega_m) t - \cos(\omega_c - 3\omega_m) t] \\
+ J_4(m_\Theta) [\cos(\omega_c + 4\omega_m) t - \cos(\omega_c - 4\omega_m) t] \\
+ ... \}$$

The $J_n(m_\Theta)$ are Bessel functions of the first kind
Spectrum of Angle Modulated Signal

Signal for PM or FM takes the form

\[ F_\Theta(t) = V_c \cos[\omega_c t + m_\Theta \sin \omega_m t] \]

From trigonometric identity,

\[ F_\Theta(t) = V_c \left[ \cos \omega_c t \cos(m_\Theta \sin \omega_m t) - \sin \omega_c t \sin(m_\Theta \sin \omega_m t) \right] \]

Consider the function

\[ g(t) = e^{-jm_\Theta \sin \omega_m t} \]

Its exponential Fourier series is represented by

\[ g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jin\omega t} \]
Spectrum of Angle Modulated Signal

Where the coefficients $c_n$ are given by

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-j\omega t} d(\omega t)$$

The Bessel function of the first kind is defined by

$$J_n(m_{\Theta}) = \frac{1}{2\pi} \int_0^{2\pi} e^{jm_{\Theta} \sin \omega_m t} \times e^{-j\omega t} d(\omega t)$$

Thus for $g(t)$, $c_n$ is given by

$$c_n = J_n(m_{\Theta})$$

And $g(t)$ can be expressed by

$$g(t) = \sum_{n=-\infty}^{\infty} J_n(m_{\Theta}) e^{j\omega t}$$
Spectrum of Angle Modulated Signal

If the \(+n\) and \(-n\) components of \(g(t)\) are summed, it can be shown that

\[
J_{-n}(m_\Theta) = (-1)^n J_n(m_\Theta)
\]

Consequently for \(n\) even

\[
J_{-n}(m_\Theta) e^{-jn\omega_m t} + J_n(m_\Theta) e^{jn\omega_m t} = J_n(m_\Theta) \left[ e^{jn\omega_m t} + e^{-jn\omega_m t} \right] = 2J_n(m_\Theta) \cos n\omega_m t
\]

For \(n\) odd, the summation yields

\[
-J_{-n}(m_\Theta) e^{-jn\omega_m t} + J_n(m_\Theta) e^{jn\omega_m t} = J_n(m_\Theta) \left[ e^{jn\omega_m t} - e^{-jn\omega_m t} \right] = j2J_n(m_\Theta) \sin n\omega_m t
\]
Spectrum of Angle Modulated Signal

Therefore \( g(t) \) can be expressed as

\[
g(t) = J_0(m_\Theta) + 2 \sum_{n \text{ even}} J_n(m_\Theta) \cos n\omega_m t \\
+ 2j \sum_{n \text{ odd}} J_n(m_\Theta) \sin n\omega_m t
\]

By application of Euler’s theorem

\[
g(t) = \cos(m_\Theta \sin \omega_m t) + j \sin(m_\Theta \sin \omega_m t)
\]

\[
\cos(m_\Theta \sin \omega_m t) = J_0(m_\Theta) + 2 \sum_{n \text{ even}} J_n(m_\Theta) \cos n\omega_m t
\]

\[
\sin(m_\Theta \sin \omega_m t) = 2 \sum_{n \text{ odd}} J_n(m_\Theta) \sin n\omega_m t
\]
Spectrum of Angle Modulated Signal

After substitution into $F_{\Theta}(t)$

$$F_{\Theta}(t) = A \cos(\omega_c t + m_{\Theta} \sin \omega_m t)$$

this yields

$$F_{\Theta}(t) = V_c \{ J_o(m_{\Theta}) \cos \omega_c t$$

$$+ J_1(m_{\Theta})[\cos(\omega_c + \omega_m)t - \cos(\omega_c - \omega_m)t]$$

$$+ J_2(m_{\Theta})[\cos(\omega_c + 2\omega_m)t - \cos(\omega_c - 2\omega_m)t]$$

$$+ J_3(m_{\Theta})[\cos(\omega_c + 3\omega_m)t - \cos(\omega_c - 3\omega_m)t]$$

$$+ J_4(m_{\Theta})[\cos(\omega_c + 4\omega_m)t - \cos(\omega_c - 4\omega_m)t]$$

$$+ ... \}$$
Bessel Functions of the First Kind

\[ J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r! \Gamma(n+r+1)} \]

\[ \Gamma(n+1) = n! \]
Zeros of Bessel Functions

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Spectrum of Angle Modulated Signal

• Selecting Modulation index
  ➢ Carrier may vanish as modulation index varies
  ➢ $\Delta f$ and $f_m$ can be chosen so as to make $J_0(m) = 0$
  ➢ Modulator can be adjusted until carrier vanishes

\[
F_{\Theta}(t) = V_c \{ J_o(m) \cos \omega_c t \\
+ J_1(m) [\cos(\omega_c + \omega_m)t - \cos(\omega_c - \omega_m)t] \\
+ J_2(m) [\cos(\omega_c + 2\omega_m)t - \cos(\omega_c - 2\omega_m)t] \\
+ J_3(m) [\cos(\omega_c + 3\omega_m)t - \cos(\omega_c - 3\omega_m)t] \\
+ J_4(m) [\cos(\omega_c + 4\omega_m)t - \cos(\omega_c - 4\omega_m)t] \\
+ \ldots \}
\]
Bandwidth of Angle Modulated Signal

Bandwidth can be approximated by

\[ BW \approx 2 \left( m_\Theta + 1 \right) f_m \approx 2 \left( \Delta f_{\text{max}} + f_m \right) \]
Spectrum of Frequency Modulated Carrier

$\beta$ is modulation index