ECE 453
Fourier Transforms

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Fourier Transforms

Fourier series is given by

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \quad \text{where} \quad T = \frac{2\pi}{\omega_o}$$

If ω_o becomes infinitely small, then the period becomes infinite and the periodic nature of the signal is lost. We then have *Fourier transforms*

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

Fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$$

Inverse Fourier transform

Properties of Fourier Transforms

	Name:	Condition:	Property:
1	Amplitude scaling	$f(t) \leftrightarrow F(\omega)$, constant K	$Kf(t) \leftrightarrow KF(\omega)$
2	Addition	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$, $\cdots$	$f(t) + g(t) + \cdots \leftrightarrow$ $F(\omega) + G(\omega) + \cdots$
3	Hermitian	Real $f(t) \leftrightarrow F(\omega)$	$F(-\omega) = F^*(\omega)$
4	Even	Real and even $f(t)$	Real and even $F(\omega)$
5	Odd	Real and odd $f(t)$	Imaginary and odd $F(\omega)$
6	Symmetry	$f(t) \leftrightarrow F(\omega)$	$F(t) \leftrightarrow 2\pi f(-\omega)$
7	Time scaling	$f(t) \leftrightarrow F(\omega)$, real c	$f(ct) \leftrightarrow \frac{1}{\lvert c \rvert} F(\frac{\omega}{c})$
8	Time shift	$f(t) \leftrightarrow F(\omega)$	$f(t - t_o) \leftrightarrow F(\omega)e^{-j\omega t_o}$
9	Frequency shift	$f(t) \leftrightarrow F(\omega)$	$f(t)e^{j\omega_o t} \leftrightarrow F(\omega - \omega_o)$
10	Modulation	$f(t) \leftrightarrow F(\omega)$	$f(t)\cos(\omega_o t) \leftrightarrow$ $\frac{1}{2}F(\omega - \omega_o) + \frac{1}{2}F(\omega + \omega_o)$
11	Time derivative	Differentiable $f(t) \leftrightarrow F(\omega)$	$\frac{df}{dt} \leftrightarrow j\omega F(\omega)$
12	Freq derivative	$f(t) \leftrightarrow F(\omega)$	$-jtf(t) \leftrightarrow \frac{d}{d\omega}F(\omega)$
13	Time convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$
14	Freq convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega)$
15	Compact form	Real $f(t)$	$f(t) =$ $\frac{1}{2\pi}\int_0^\infty 2\lvert F(\omega)\rvert \cos(\omega t +$ $\angle F(\omega))d\omega$
16	Parseval, Energy W	$f(t) \leftrightarrow F(\omega)$	$W \equiv \int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt =$ $\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2 d\omega$

*From Kudeki – Analog Signals and Systems

Fourier Transform Pairs - Table

	$f(t) \leftrightarrow F(\omega)$		
1	$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega},\ a>0$	14	$\delta(t) \leftrightarrow 1$
2	$e^{at}u(-t) \leftrightarrow \frac{1}{a-j\omega},\ a>0$	15	$1 \leftrightarrow 2\pi\delta(\omega)$
3	$e^{-a\lvert t\rvert} \leftrightarrow \frac{2a}{a^2+\omega^2},\ a>0$	16	$\delta(t-t_o) \leftrightarrow e^{-j\omega t_o}$
4	$\frac{a^2}{a^2+t^2} \leftrightarrow \pi a e^{-a\lvert\omega\rvert},\ a>0$	17	$e^{j\omega_o t} \leftrightarrow 2\pi\delta(\omega-\omega_o)$
5	$te^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^2},\ a>0$	18	$\cos(\omega_o t) \leftrightarrow \pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]$
6	$t^n e^{-at}u(t) \leftrightarrow \frac{n!}{(a+j\omega)^{n+1}},\ a>0$	19	$\sin(\omega_o t) \leftrightarrow j\pi[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)]$
7	$\mathrm{rect}(\frac{t}{\tau}) \leftrightarrow \tau\,\mathrm{sinc}(\frac{\omega\tau}{2})$	20	$\cos(\omega_o t)u(t) \leftrightarrow$ $\frac{\pi}{2}[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]+\frac{j\omega}{\omega_o^2-\omega^2}$
8	$\mathrm{sinc}(Wt) \leftrightarrow \frac{\pi}{W}\mathrm{rect}(\frac{\omega}{2W})$	21	$\sin(\omega_o t)u(t) \leftrightarrow$ $j\frac{\pi}{2}[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)]+\frac{\omega_o}{\omega_o^2-\omega^2}$
9	$\Delta(\frac{t}{\tau}) \leftrightarrow \frac{\tau}{2}\mathrm{sinc}^2(\frac{\omega\tau}{4})$	22	$\mathrm{sgn}(t) \leftrightarrow \frac{2}{j\omega}$
10	$\mathrm{sinc}^2(\frac{Wt}{2}) \leftrightarrow \frac{2\pi}{W}\Delta(\frac{\omega}{2W})$	23	$u(t) \leftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$
11	$e^{-at}\sin(\omega_o t)u(t) \leftrightarrow$ $\frac{\omega_o}{(a+j\omega)^2+\omega_o^2},\ a>0$	24	$\sum_{n=-\infty}^{\infty}\delta(t-nT) \leftrightarrow \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega-n\frac{2\pi}{T})$
12	$e^{-at}\cos(\omega_o t)u(t) \leftrightarrow$ $\frac{a+j\omega}{(a+j\omega)^2+\omega_o^2},\ a>0$	25	$\sum_{n=-\infty}^{\infty} f(t)\delta(t-nT) \leftrightarrow$ $\sum_{n=-\infty}^{\infty}\frac{1}{T}F(\omega-n\frac{2\pi}{T})$
13	$e^{-\frac{t^2}{2\sigma^2}} \leftrightarrow \sigma\sqrt{2\pi}\,e^{-\frac{\sigma^2\omega^2}{2}}$		

*From Kudeki – Analog Signals and Systems