ECE 453
Single Sideband Modulation

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Causality Principle

Consider a function $h(t)$

$$h(t) = 0, \quad t < 0$$

Every function can be considered as the sum of an even function and an odd function

$$h(t) = h_e(t) + h_o(t)$$

For an even function $h_e(t)$,

$$h_e(t) = \frac{1}{2} [h(t) + h(-t)]$$

For an odd function $h_o(t)$,

$$h_o(t) = \frac{1}{2} [h(t) - h(-t)]$$

$$h_o(t) = \begin{cases} h_e(t), & t > 0 \\ -h_e(t), & t < 0 \end{cases}$$

$$h_o(t) = \text{sgn}(t)h_e(t)$$
Hilbert Transform

\[ h(t) = h_e(t) + \text{sgn}(t)h_e(t) \]

In frequency domain this becomes

\[ H(f) = H_e(f) + \frac{1}{j\pi f} * H_e(f) \]

\[ H(f) = H_e(f) - j\hat{H}_e(f) \]

\( \hat{H}_e(f) \) is the Hilbert transform of \( H_e(f) \)

Make use of

\[ \text{sgn}(t) \leftrightarrow \frac{1}{j\pi f} \]

\( \hat{H}(f) = H(f) * \frac{1}{\pi f} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{H(\xi)}{f - \xi} \, d\xi \)

\( \Rightarrow \) Imaginary part of transfer function is related to the real part through the Hilbert transform
Hilbert Transform

\[ m(t) \leftrightarrow M(\omega) \]
\[ \text{sgn}(t)m(t) \leftrightarrow -j\hat{M}(\omega) \]

Using symmetry property of Fourier transforms

\[ f(t) \leftrightarrow F(\omega) \Rightarrow F(t) \leftrightarrow 2\pi f(-\omega) \]

\[ \hat{m}(t) \leftrightarrow -jM(\omega)\text{sgn}(\omega) \]
\[ \hat{m}(t) \]

is Hilbert transform of
\[ m(t) \]
Hermitian Property

\[ \text{Real } f(t) \iff F(\omega) \implies F(-\omega) = F^*(\omega) \]
Modulation Theorem

\[ A(t) \cos(\omega_c t + \theta) \leftrightarrow \frac{1}{2} e^{j\theta} A(\omega - \omega_c) + \frac{1}{2} e^{-j\theta} A(\omega + \omega_c) \]

The spectrum of the modulated signal can be obtained by superimposing two copies of the spectrum of \(A(t)\) that have been displaced by \(+\omega_c\) and \(-\omega_c\) on the frequency axis.
Motivation for SSB

Amplitude modulation and DSB-SC techniques require transmission bandwidth of twice the bandwidth of the modulating signal $m(t)$.

In both cases the transmission bandwidth is occupied by the upper sideband (USB) and lower sideband (LSB)

- **Observations on SSB**
  - USB and LSB are uniquely related to each other, as they are symmetric with respect to $f_c$.
  - Therefore, it is enough to transmit only one side band.
  - For demodulation SSB can be coherently demodulated by multiplying with $\cos(\omega_c t)$ and followed by LPF.
SSB - Frequency Domain Representation

- M(ω)
- DSB-SC
- SSB₁ (USB)
- SSB₂ (LSB)
SSB - Time Domain Representation

\[ m(t) \leftrightarrow M(f) \]

Baseband modulating signal (real)

\[ m_+(t) \leftrightarrow M_+(f) \]

USB signal

\[ m_-(t) \leftrightarrow M_-(f) \]

LSB signal
SSB - Time Domain Representation

\[ M_-(\omega + \omega_c) \]
\[ M_+(\omega - \omega_c) \]

\[ M_+(\omega + \omega_c) \]
\[ M_-(\omega - \omega_c) \]
SSB – Hilbert Transform

Using the spectrum relationship

\[
M_+(f) = M(f)u(f) = M(f) \frac{1}{2} [1 + \text{sgn}(f)] = \frac{1}{2} [M(f) + j\bar{M}(f)]
\]

\[
M_-(f) = M(f)u(-f) = M(f) \frac{1}{2} [1 - \text{sgn}(f)] = \frac{1}{2} [M(f) - j\bar{M}(f)]
\]

where

\[
\frac{1}{2} j\bar{M}(f) = \frac{1}{2} M(f) \text{sgn}(f)
\]

\[
\bar{M}(f) = M(f) \times [-j \text{sgn}(f)]
\]
SSB – Hilbert Transform

We have \( \tilde{M}(f) = M(f) \times [-j \text{sgn}(f)] \)

The Fourier series pair \( \text{sgn}(t) \leftrightarrow \frac{1}{j\pi f} \)

\[
\hat{m}(t) = m(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{m(\tau)}{t-\tau} d\tau
\]

\( \hat{m}(t) \leftrightarrow \tilde{M}(f) \)

Thus, \( \hat{m}(t) \) is the Hilbert transform of \( m(t) \)
SSB – Hilbert Transform

\[ H(f) = -j \text{sgn}(f) = \begin{cases} 
-j & f \geq 0 \\
+j & f < 0 
\end{cases} \]

\( H(f) \): wideband phase shifter (Hilbert Transform)
SSB – Hilbert Transform

By delaying the phase of every component of \( m(t) \) by \( \pi/2 \) we get \( \hat{m}(t) \) the Hilbert transform of \( m(t) \)

⇒ Hilbert transformer is ideal phase shifter

\[
m_+(t) = \frac{1}{2} [m(t) + j\hat{m}(t)]
\]

\[
m_-(t) = \frac{1}{2} [m(t) - j\hat{m}(t)]
\]

\( \hat{m}(t) \) is the Hilbert transform of \( m(t) \)
SSB – Time-Domain Representation

\[ S_{USB}(f) = M_+(f - f_c) + M_-(f + f_c) \]

\[ S_{USB}(f) = \frac{1}{2} \left[ M(f - f_c) + M(f + f_c) \right] + \frac{1}{2j} \left[ \bar{M}(f - f_c) - \bar{M}(f + f_c) \right] \]

The inverse Fourier transform is then

\[ s_{USB}(t) = m(t) \cos(\omega_c t) - \hat{m}(t) \sin(\omega_c t) \]

Similarly, we can show that

\[ s_{LSB}(t) = m(t) \cos(\omega_c t) + \hat{m}(t) \sin(\omega_c t) \]

So, in general, we have

\[ s_{SSB}(t) = m(t) \cos(\omega_c t) \mp \hat{m}(t) \sin(\omega_c t) \text{ (USB and LSB)} \]
Generation of SSB Signals

• **Selective Filtering Method**
  - Use $m(t)$ to generate DSB-SC ($m(t) \cos \omega_c t$)
  - Feed DSB-SC through a band-pass filter (BPF)
  - We must have $B << f_c$
  - $m(t)$ must have little or no frequency content at DC
Generation of SSB Signals

Gap between sidebands must exist
Phase-shift Method

\[ s_{SSB}(t) = m(t) \cos(\omega_c t) \mp \hat{m}(t) \sin(\omega_c t) \]
Demodulation of SSB Signals

Any DSB-SC coherent demodulation technique can be used

\[ s_{SSB}(t) \cos(\omega_c t) = m(t) \cos^2(\omega_c t) \mp \hat{m}(t) \sin(\omega_c t) \cos(\omega_c t) \]

\[ = \frac{1}{2} m(t) + \frac{1}{2} m(t) \cos(2\omega_c t) \pm \frac{1}{2} \hat{m}(t) \sin(2\omega_c t) \]

If we filter signal with a LPF, we can eliminate components centered at \(2f_c\) and filter output will be \(\sim m(t)\)
Demodulation of SSB Signals

upper sideband (USB)

Demodulator

C

LPF

km(t)

carrier

Input

Carrier

f_c
Demodulation of SSB Signals

-2f_c

2f_c

-2f_c

2f_c