ECE 453
Single Sideband Modulation

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Causality Principle

Consider a function *h(t)*

$$h(t) = 0, \quad t < 0$$

Every function can be considered as the sum of an even function and an odd function

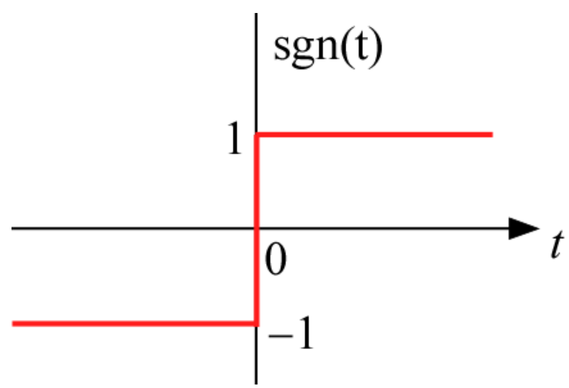
$$h(t) = h_e(t) + h_o(t)$$

$$h_e(t) = \frac{1}{2}\left[h(t) + h(-t)\right]$$ **Even function**

$$h_o(t) = \frac{1}{2}\left[h(t) - h(-t)\right]$$ **Odd function**

$$h_o(t) = \begin{cases} h_e(t), & t > 0 \\ -h_e(t), & t < 0 \end{cases}$$

$$h_o(t) = \text{sgn}(t) h_e(t)$$



Hilbert Transform

$$h(t) = h_e(t) + \text{sgn}(t)h_e(t)$$

In frequency domain this becomes

$$H(f) = H_e(f) + \frac{1}{j\pi f} * H_e(f)$$

Make use of

$$\text{sgn}(t) \leftrightarrow \frac{1}{j\pi f}$$

$$H(f) = H_e(f) - j\hat{H}_e(f)$$

$\hat{H}_e(f)$ is the Hilbert transform of $H_e(f)$

$$\hat{H}(f) = H(f) * \frac{1}{\pi f} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{H(\xi)}{f-\xi} d\xi$$

➔**Imaginary part of transfer function is related to the real part through the Hilbert transform**

Hilbert Transform

$$m(t) \leftrightarrow M(\omega)$$

$$\text{sgn}(t)m(t) \leftrightarrow -j\hat{M}(\omega)$$

$\hat{M}(\omega)$ *is Hilbert transform of* $M(\omega)$

Using symmetry property of Fourier transforms

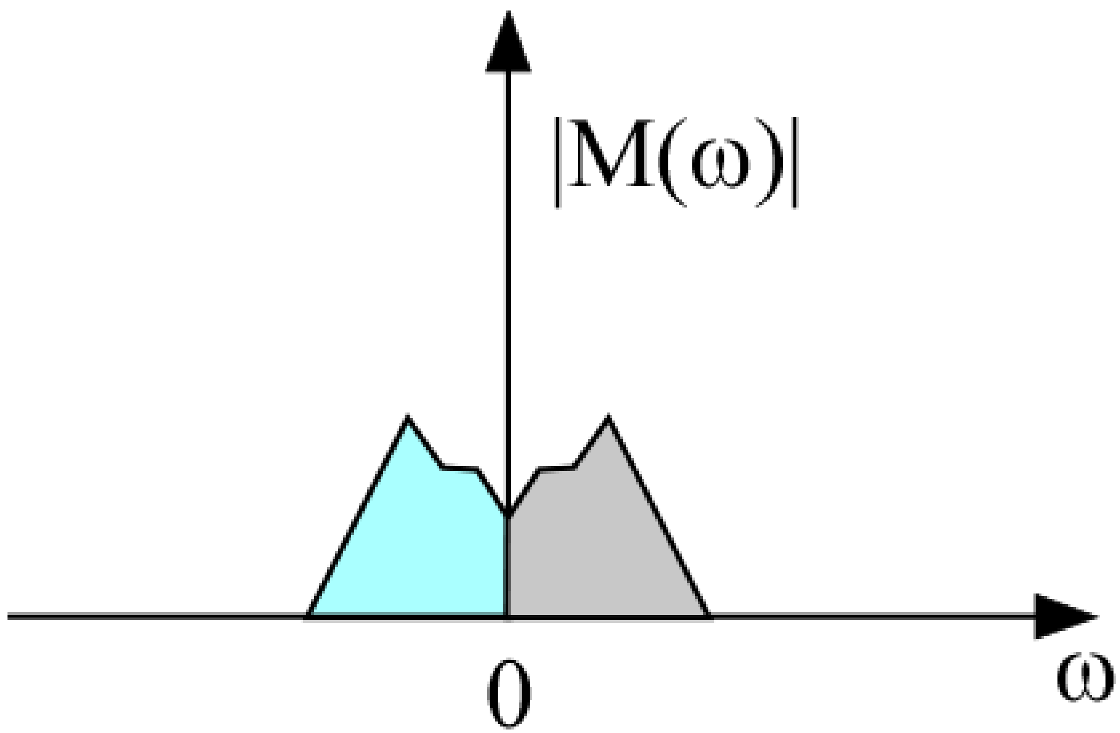
$$f(t) \leftrightarrow F(\omega) \Rightarrow F(t) \leftrightarrow 2\pi f(-\omega)$$

$$\hat{m}(t) \leftrightarrow -jM(\omega)\text{sgn}(\omega)$$

$\hat{m}(t)$ *is Hilbert transform of* $m(t)$

Hermitian Property

Real $f(t) \leftrightarrow F(\omega) \Rightarrow F(-\omega) = F^*(\omega)$



Modulation Theorem

$$A(t)\cos(\omega_c t+\theta) \leftrightarrow \frac{1}{2}e^{j\theta}A\left(\omega-\omega_c\right)+\frac{1}{2}e^{-j\theta}A\left(\omega+\omega_c\right)$$

The spectrum of the modulated signal can be obtained by superimposing two copies of the spectrum of *A(t)* that have been displaced by $+\omega_c$ and $-\omega_c$ on the frequency axis.

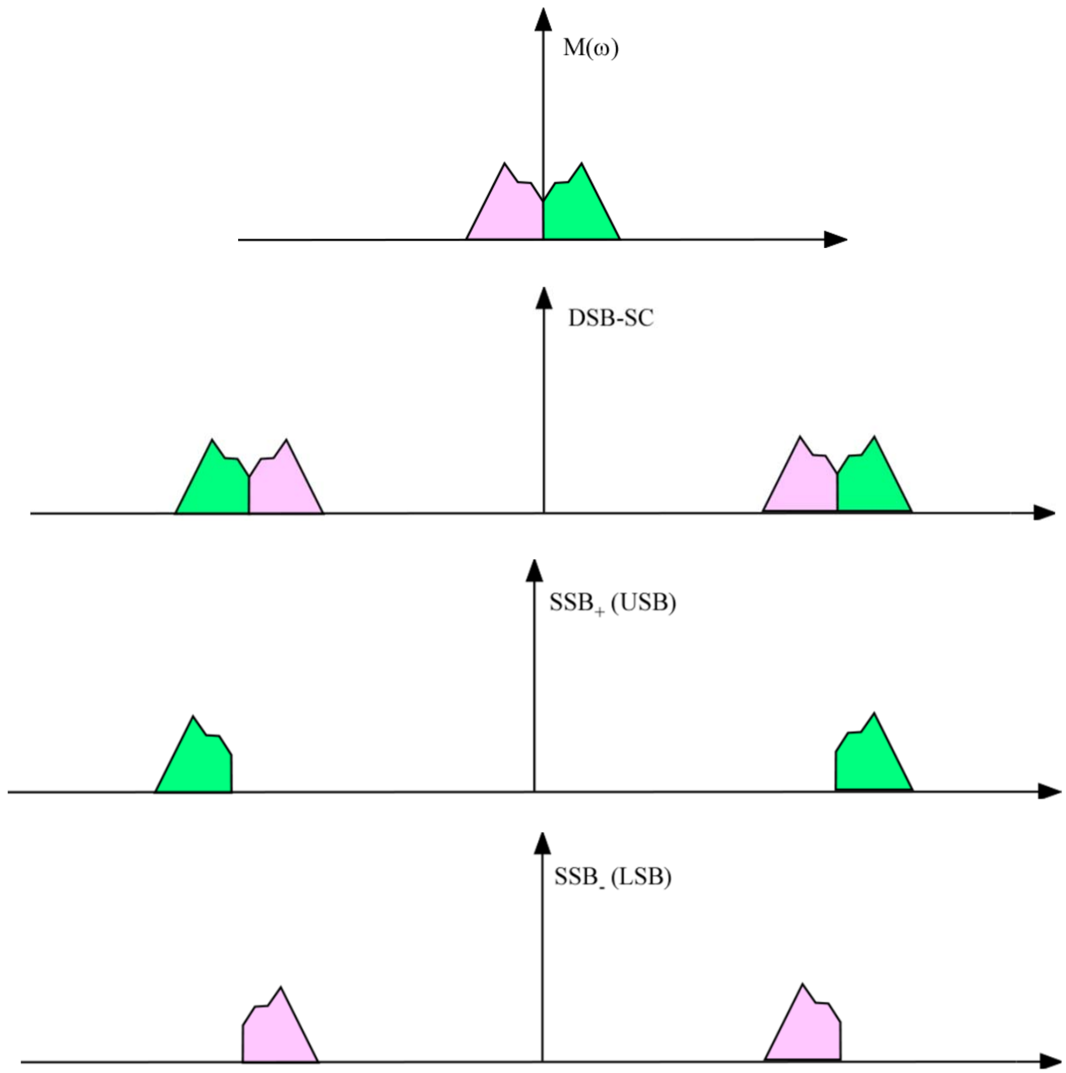
Motivation for SSB

Amplitude modulation and DSB-SC techniques require transmission bandwidth of twice the bandwidth of the modulating signal *m(t)*.

In both cases the transmission bandwidth is occupied by the upper sideband (USB) and lower sideband (LSB)

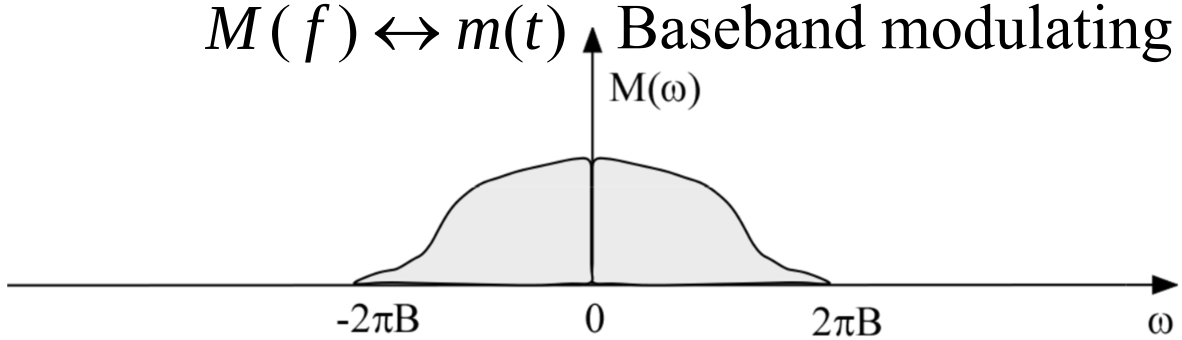
- **Observations on SSB**
  - USB and LSB are uniquely related to each other, as they are symmetric with respect to f_c.
  - Therefore, it is enough to transmit only one side band.
  - For demodulation SSB can be coherently demodulated by multiplying with $cos(\omega_c t)$ and followed by LPF.

SSB - Frequency Domain Representation

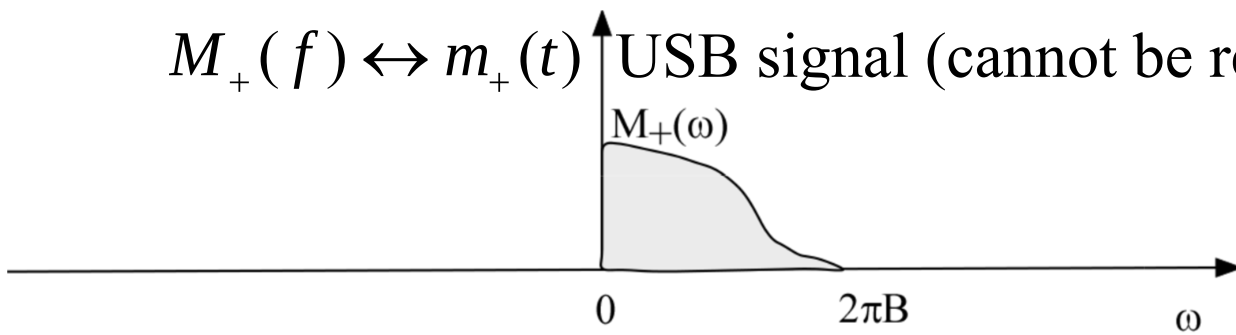


SSB - Time Domain Representation

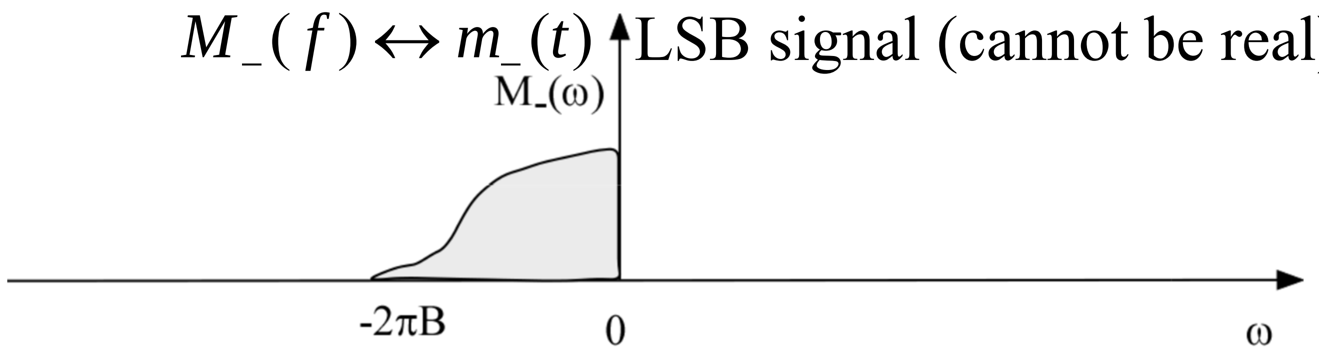
$M(f) \leftrightarrow m(t)$ Baseband modulating signal (real)



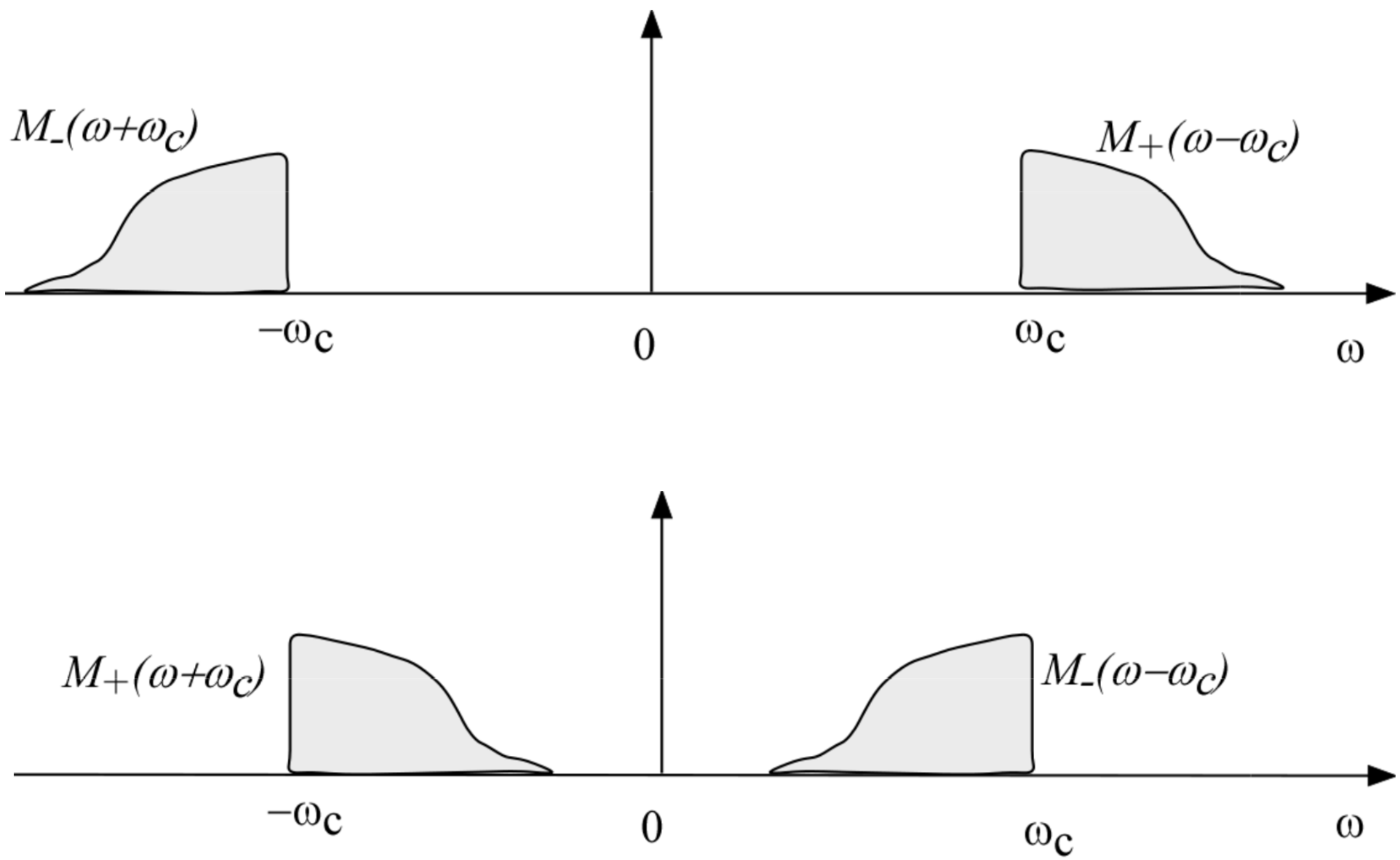
$M_+(f) \leftrightarrow m_+(t)$ USB signal (cannot be real)



$M_-(f) \leftrightarrow m_-(t)$ LSB signal (cannot be real)



SSB - Time Domain Representation



SSB – Hilbert Transform

Using the spectrum relationship

$$M_+(f) = M(f)u(f) = M(f)\frac{1}{2}\left[1+\text{sgn}(f)\right] = \frac{1}{2}\left[M(f) + j\bar{M}(f)\right]$$

$$M_-(f) = M(f)u(-f) = M(f)\frac{1}{2}\left[1-\text{sgn}(f)\right] = \frac{1}{2}\left[M(f) - j\bar{M}(f)\right]$$

where

$$\frac{1}{2}j\bar{M}(f) = \frac{1}{2}M(f)\text{sgn}(f)$$

$$\bar{M}(f) = M(f)\times\left[-j\,\text{sgn}(f)\right]$$

SSB – Hilbert Transform

We have $\bar{M}(f) = M(f) \times [-j\,\mathrm{sgn}(f)]$

The Fourier series pair $-j\,\mathrm{sgn}(f) \leftrightarrow \dfrac{1}{\pi t}$

$$\hat{m}(t) = m(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{m(\tau)}{t - \tau} d\tau$$

$$\hat{m}(t) \leftrightarrow \bar{M}(f)$$

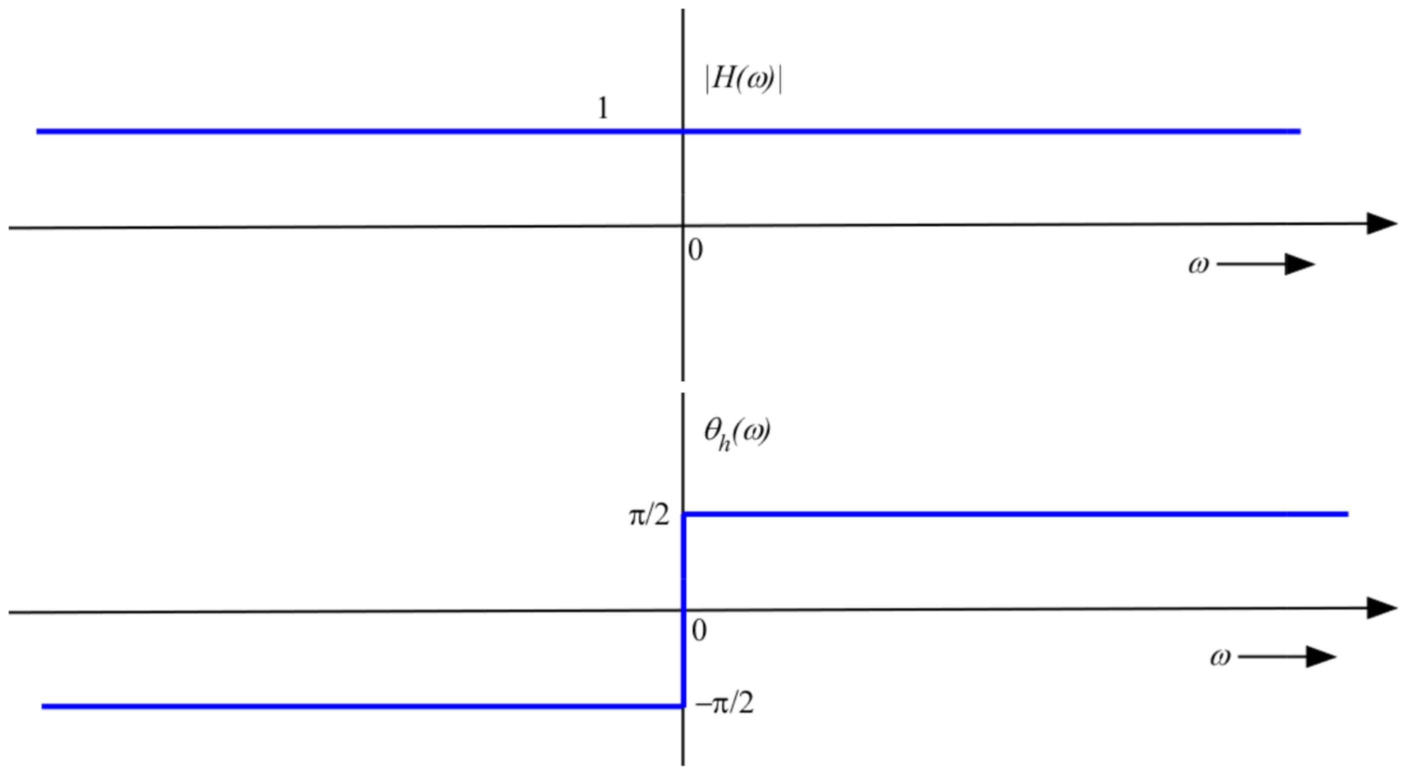
Thus, the inverse Fourier transform of $\bar{M}(f)$

is the Hilbert transform of *m(t)*

SSB – Hilbert Transform

$$H(f) = -j\,\mathrm{sgn}(f) = \begin{cases} -j & f \geq 0 \\ j & f < 0 \end{cases}$$

H(f): wideband phase shifter (Hilbert Transform)



SSB – Hilbert Transform

By delaying the phase of every component of *m(t)* by *π/2* we get $\hat{m}(t)$ the Hilbert transform of *m(t)*

➔ **Hilbert transformer is ideal phase shifter**

$$m_+(t) = \frac{1}{2}\left[m(t) + j\hat{m}(t)\right]$$

$$m_-(t) = \frac{1}{2}\left[m(t) - j\hat{m}(t)\right]$$

$\hat{m}(t)$ is the Hilbert transform of *m(t)*

SSB – Time-Domain Representation

$$S_{USB}(f) = M_{+}(f - f_c) + M_{-}(f + f_c)$$

$$S_{USB}(f) = \frac{1}{2}\left[M(f - f_c) + M(f + f_c)\right] + \frac{1}{2j}\left[\bar{M}(f - f_c) - \bar{M}(f + f_c)\right]$$

The inverse Fourier transform is then

$$s_{USB}(t) = m(t)\cos\left(\omega_c t\right) - \hat{m}(t)\sin\left(\omega_c t\right)$$

Similarly, we can show that

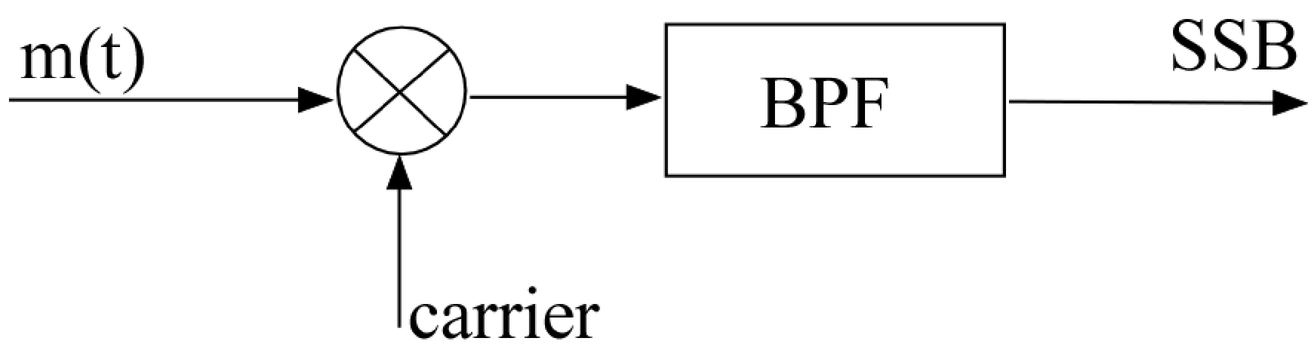
$$s_{LSB}(t) = m(t)\cos\left(\omega_c t\right) + \hat{m}(t)\sin\left(\omega_c t\right)$$

So, in general, we have

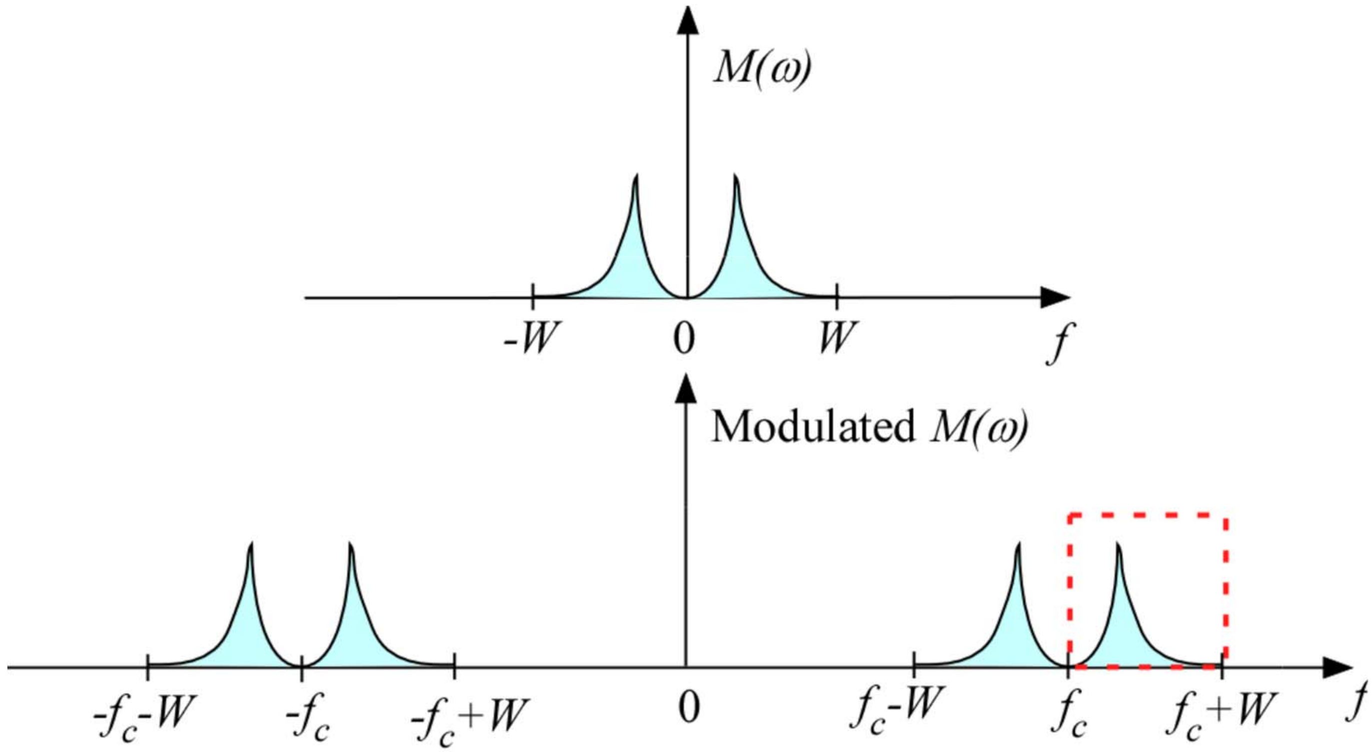
$$s_{SSB}(t) = m(t)\cos\left(\omega_c t\right) \mp \hat{m}(t)\sin\left(\omega_c t\right)$$ **(USB and LSB)**

Generation of SSB Signals

- **Selective Filtering Method**
  - Use *m(t)* to generate DSB-SC ($m(t)\cos\omega_c t$)
  - Feed DSB-SC through a band-pass filter (BPF)
  - We must have $B << f_c$
  - *m(t)* must have little or no frequency content at DC



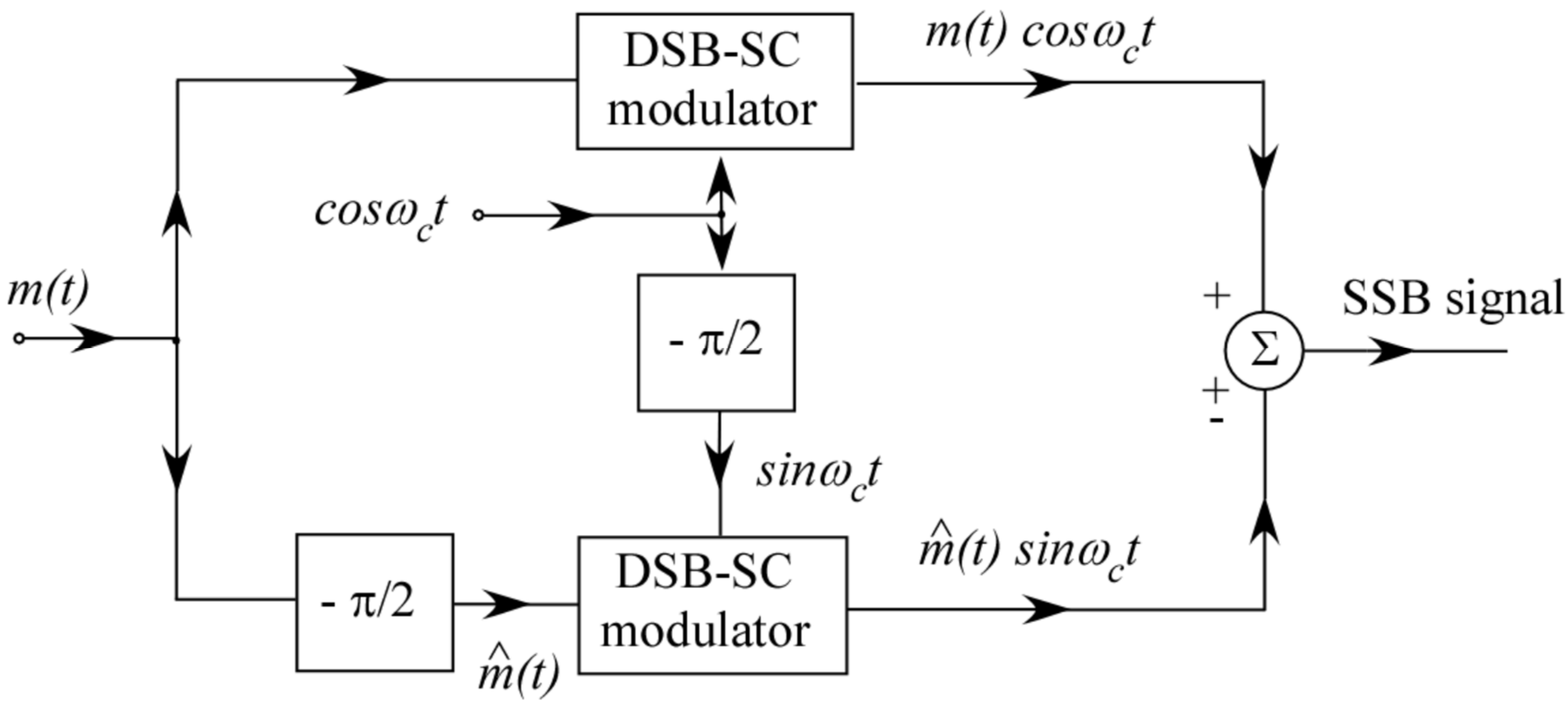
Generation of SSB Signals



Gap between sidebands must exist

Phase-shift Method

$$s_{SSB}(t) = m(t)\cos\left(\omega_c t\right) \mp \hat{m}(t)\sin\left(\omega_c t\right)$$



Demodulation of SSB Signals

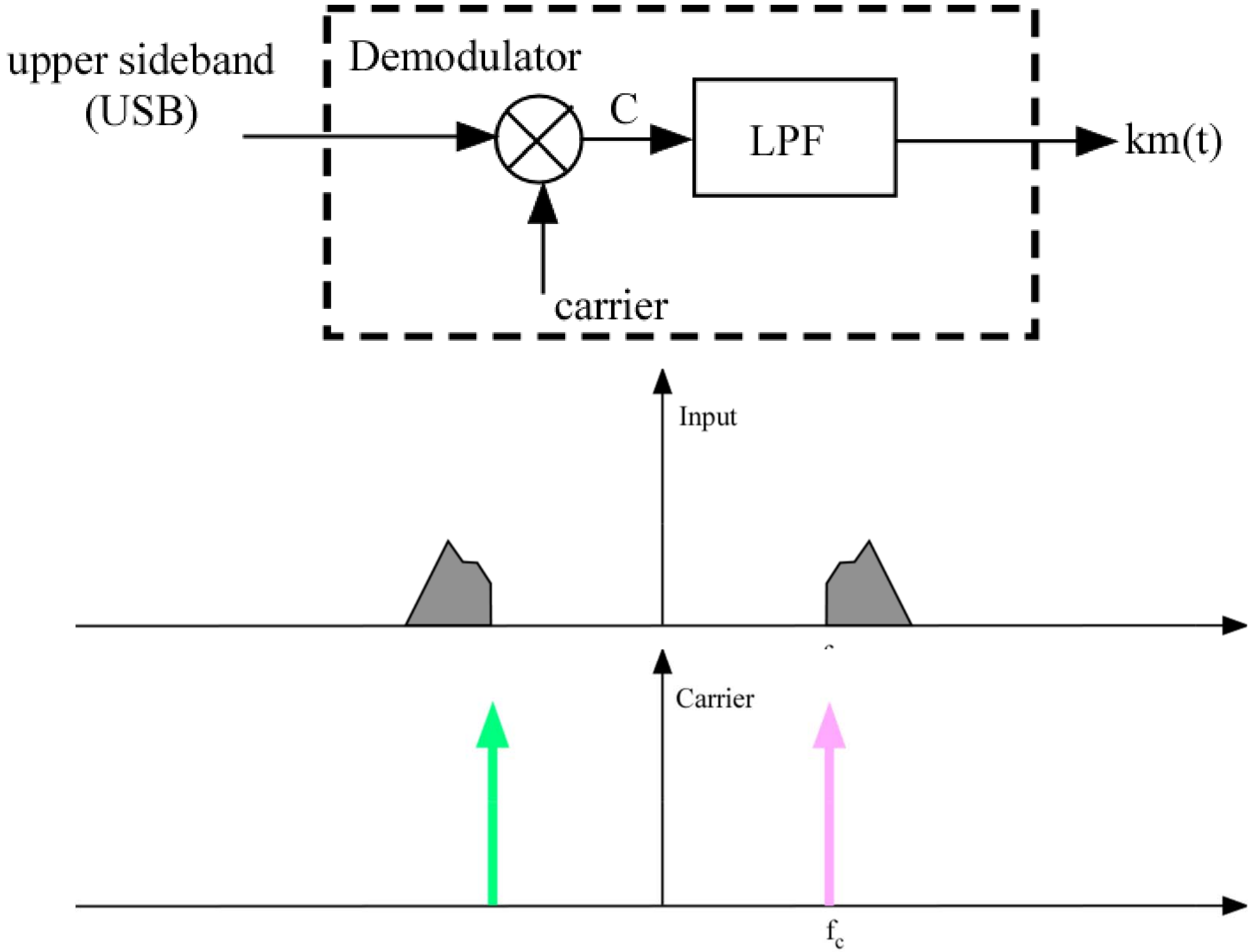
Any DSB-SC coherent demodulation technique can be used

$$s_{SSB}(t)\cos(\omega_c t) = m(t)\cos^2(\omega_c t) \mp \hat{m}(t)\sin(\omega_c t)\cos(\omega_c t)$$

$$= \frac{1}{2}m(t) + \frac{1}{2}m(t)\cos(2\omega_c t) \pm \frac{1}{2}\hat{m}(t)\sin(2\omega_c t)$$

If we filter signal with a LPF, we can eliminate components centered at $2f_c$ and filter output will be ~ *m(t)*

Demodulation of SSB Signals



Demodulation of SSB Signals