ECE 546
Lecture 02
Review of Electromagnetics

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Electromagnetic Quantities

$\vec{E}$ **Electric field (Volts/m)**

$\vec{D}$ **Electric flux density (Coulombs/m^2)**

$\vec{H}$ **Magnetic field (Amperes/m)**

$\vec{B}$ **Magnetic flux density (Webers/m^2)**

$\vec{J}$ **Current density (Amperes/m^2)**

ρ **Charge density (Coulombs/m^2)**

Maxwell's Equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law of Induction

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Ampère's Law

$$\nabla \cdot \vec{D} = \rho$$

Gauss' Law for electric field

$$\nabla \cdot \vec{B} = 0$$

Gauss' Law for magnetic field

Constitutive Relations

$$\vec{B} = \mu \vec{H} \qquad \qquad \vec{D} = \varepsilon \vec{E}$$

Permittivity ε : Farads/m

Permeability μ : Henries/m

Free Space

$$\varepsilon_o = 8.85 \times 10^{-12} \quad F/m$$

$$\mu_o = 4\pi \times 10^{-7} \quad H/m$$

Continuity Equation

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{D}}{\partial t} = \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \nabla \cdot \vec{D} = 0$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Electrostatics

Assume no time dependence $\frac{\partial}{\partial t} \rightarrow 0$

$$\nabla \times \vec{E} = 0$$

Since $\nabla \times \vec{E} = 0$, $\exists \phi$ such that

$\vec{E} = -\nabla \phi$ where ϕ is the scalar potential

with $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$ we get $\nabla \cdot \left(-\nabla \phi\right) = -\nabla^2 \phi = \frac{\rho}{\varepsilon}$

we get $\nabla \cdot \left(-\nabla \phi\right) = -\nabla^2 \phi = \frac{\rho}{\varepsilon}$ ← **Poisson's Equation**

if no charge is present $\nabla^2 \phi = 0$ ← **Laplace's Equation**

Integral Form of ME

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{S}$$

Faraday's Law of Induction

$$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S} + \frac{\partial}{\partial t}\int_S \vec{D}\cdot d\vec{S}$$

Ampere's Law

$$\oint_S \vec{D}\cdot d\vec{S} = \int_V \rho\cdot dv$$

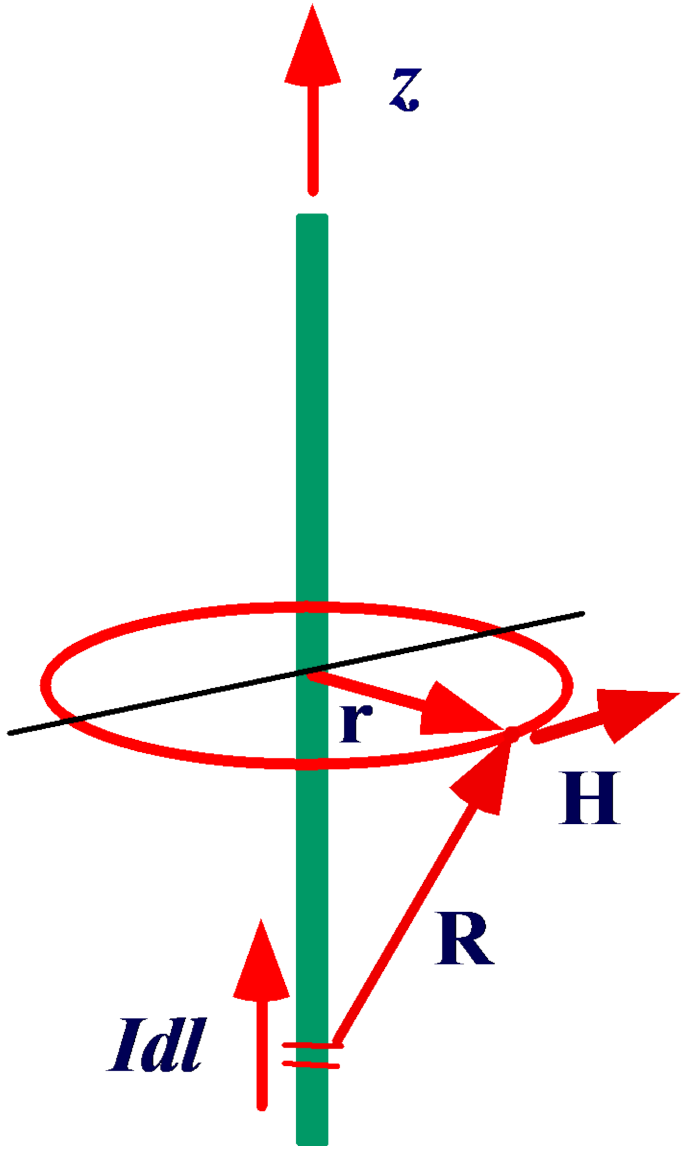
Gauss' Law – D vector

$$\oint_S \vec{B}\cdot d\vec{S} = 0$$

Gauss' Law – B Field

H-Field – Infinite Length Wire

- Use azimuthal symmetry
- Assume return path is at infinity
- Use Stokes' theorem

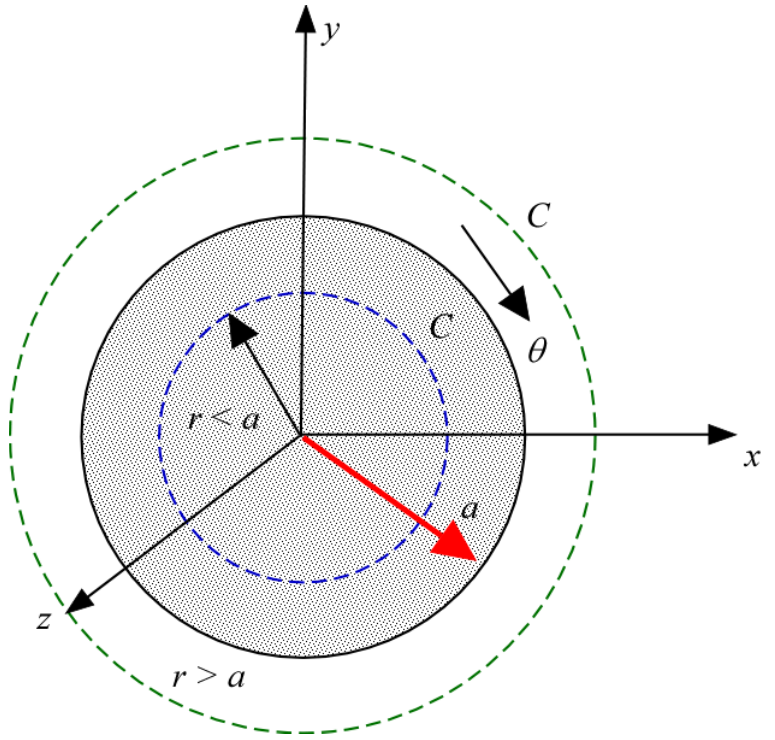


Using Gauss' Law – D Field

Spherical volume of radius a with uniform charge density ρ_o(C/m³)

$$\oint_S \vec{D}\cdot d\vec{S} = \int_V \rho \cdot dv$$

$$\oint_S \vec{D}\cdot d\vec{S} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D_r(r)\hat{r}\cdot r^2 \sin\theta d\theta d\phi \hat{r}$$



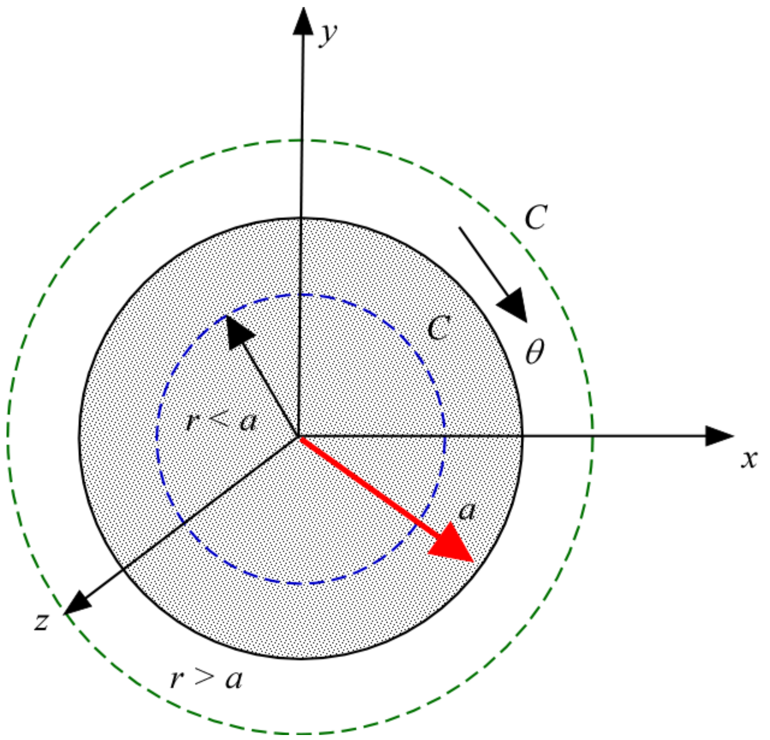
Charge exists only for $r < a$ and with uniform density

$$\int_V \rho \cdot dv = \begin{cases} \frac{4}{3}\pi r^3 \rho_o & \text{for } r \le a \\ \frac{4}{3}\pi a^3 \rho_o & \text{for } r \ge a \end{cases}$$

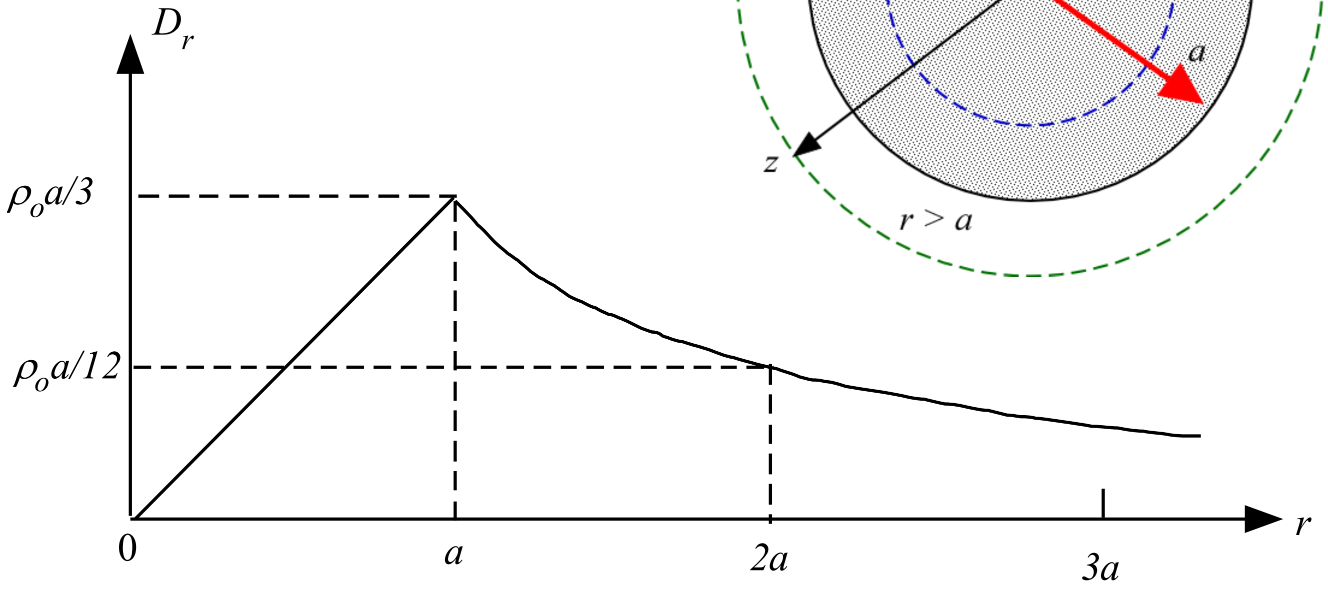
Using Gauss' Law – D Field

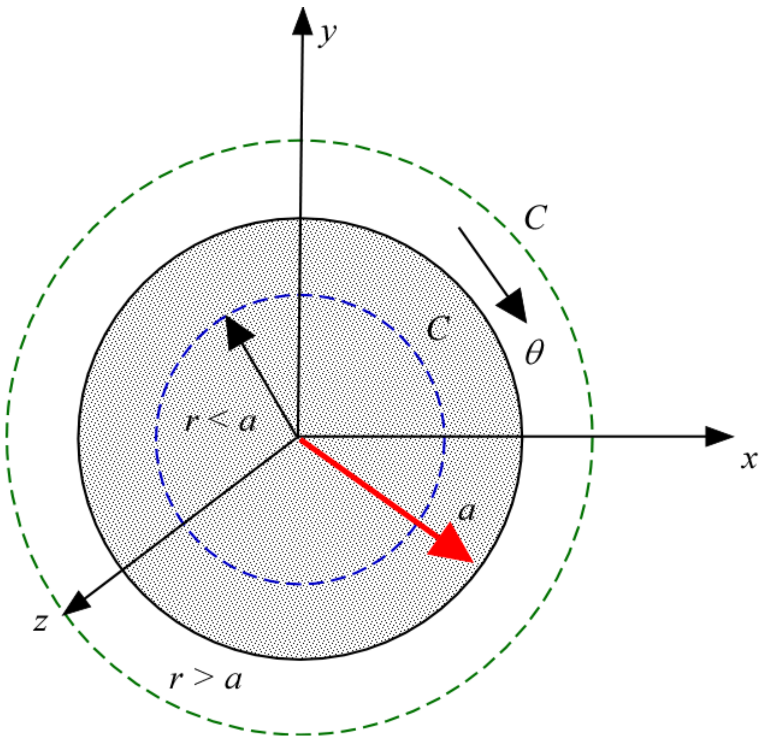
$$4\pi r^2 D_r(r) = \begin{cases} \dfrac{4}{3}\pi r^3 \rho_o & \text{for } r \le a \\ \dfrac{4}{3}\pi a^3 \rho_o & \text{for } r \ge a \end{cases}$$

$$D_r(r) = \begin{cases} \dfrac{\rho_o r}{3} & \text{for } r \le a \\ \dfrac{\rho_o a^3}{3r^2} & \text{for } r \ge a \end{cases}$$



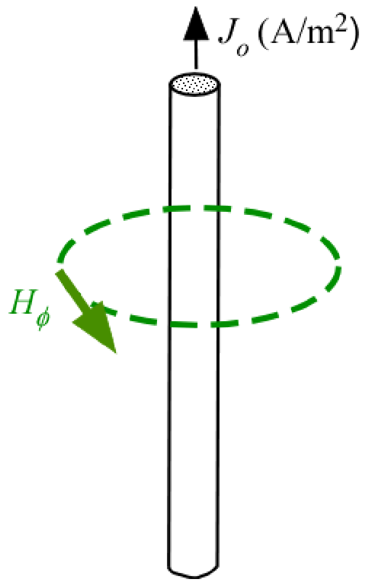
Using Gauss' Law – D Field



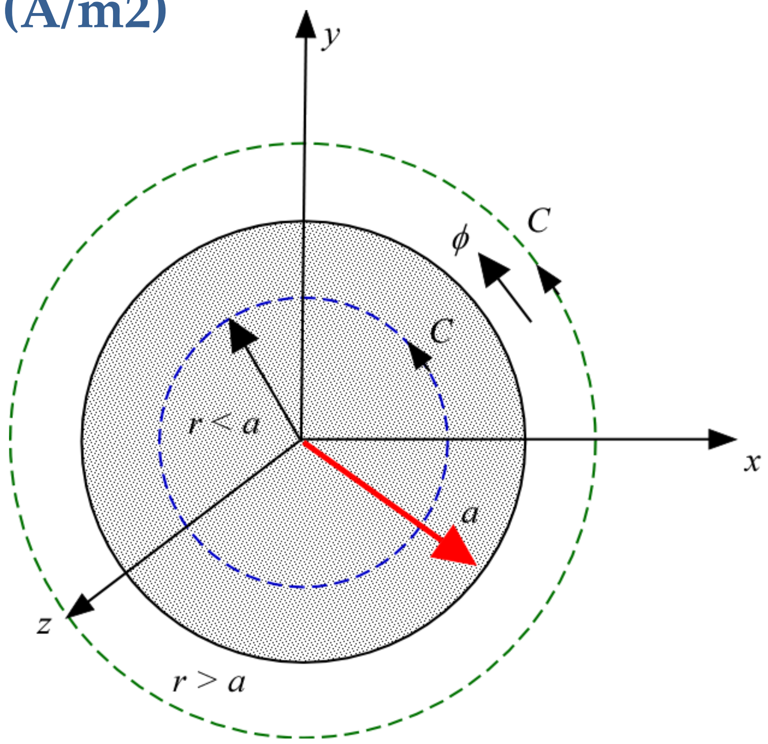


Using Gauss' Law – H Field

Infinitely long cylindrical wire of radius a with uniform current density J_o (A/m2)



wire



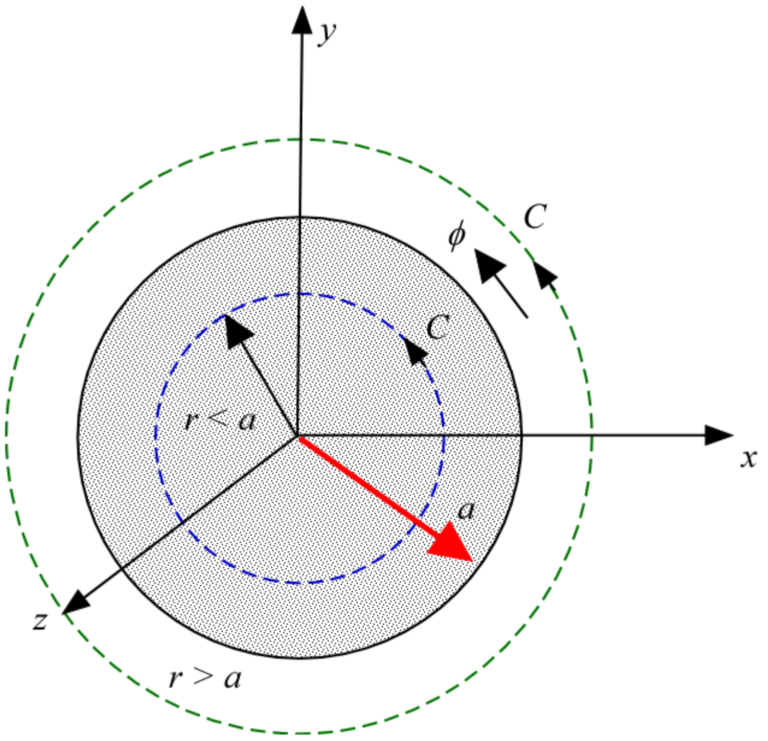
cross section

Need to calculate H field both inside and outside wire ➔ Use azimuthal symmetry

Using Gauss' Law – H Field

$$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S} + \frac{\partial}{\partial t}\int_S \vec{D}\cdot d\vec{S}$$

$$\oint_C \vec{H}\cdot d\vec{l} = \int_{\phi=0}^{2\pi} H_\phi(r)\hat{\phi}\cdot r d\phi\hat{\phi} = 2\pi r H_\phi(r)$$



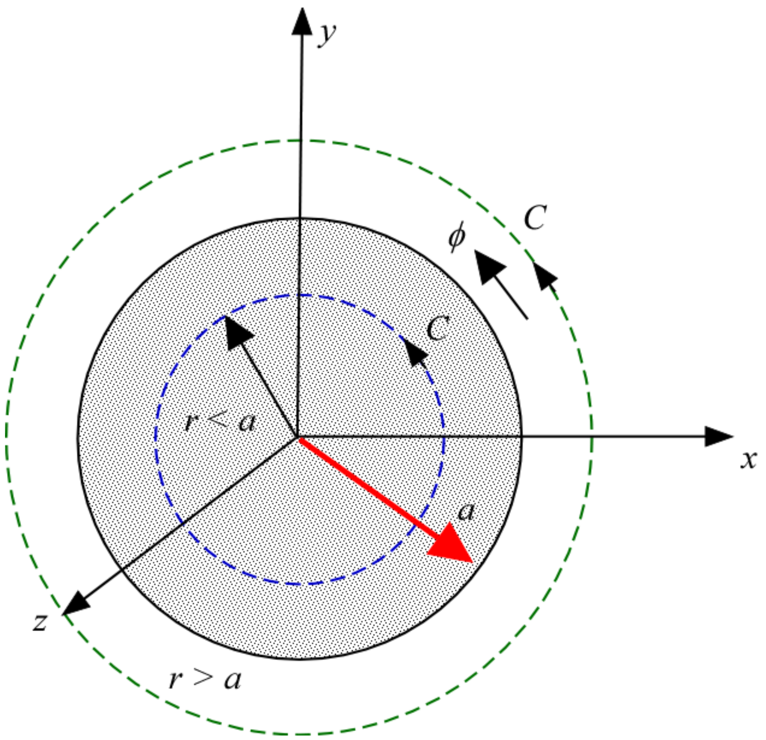
Current exists only for $r < a$ and with uniform density

$$\int_S \vec{J}\cdot d\vec{S} = \begin{cases} \int_{r=0}^{r}\int_{\phi=0}^{2\pi} J_o \hat{z}\cdot r\,dr\,d\phi\,\hat{z} & for\ r \le a \\ \int_{r=0}^{a}\int_{\phi=0}^{2\pi} J_o \hat{z}\cdot r\,dr\,d\phi\,\hat{z} & for\ r \ge a \end{cases}$$

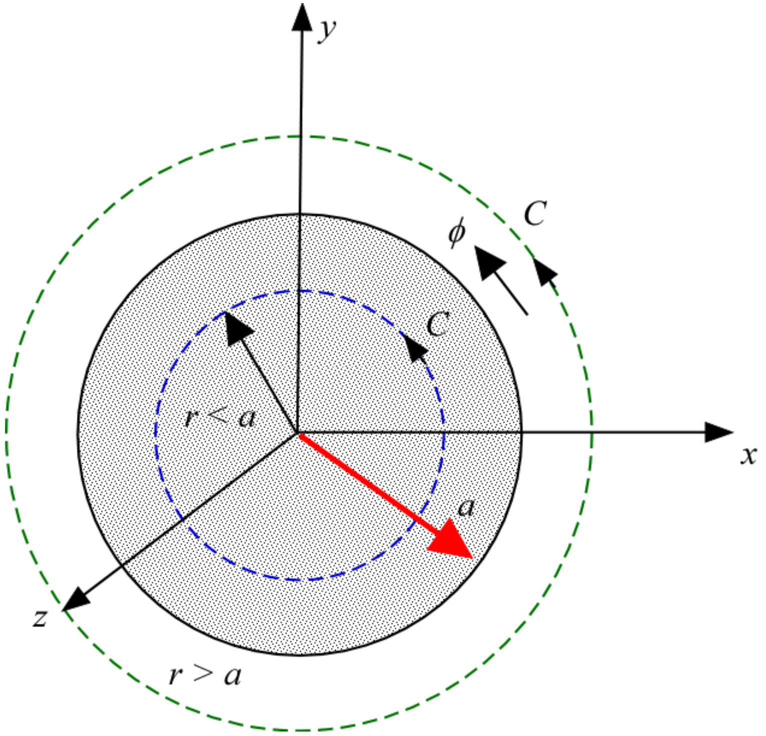
Using Gauss' Law – H Field

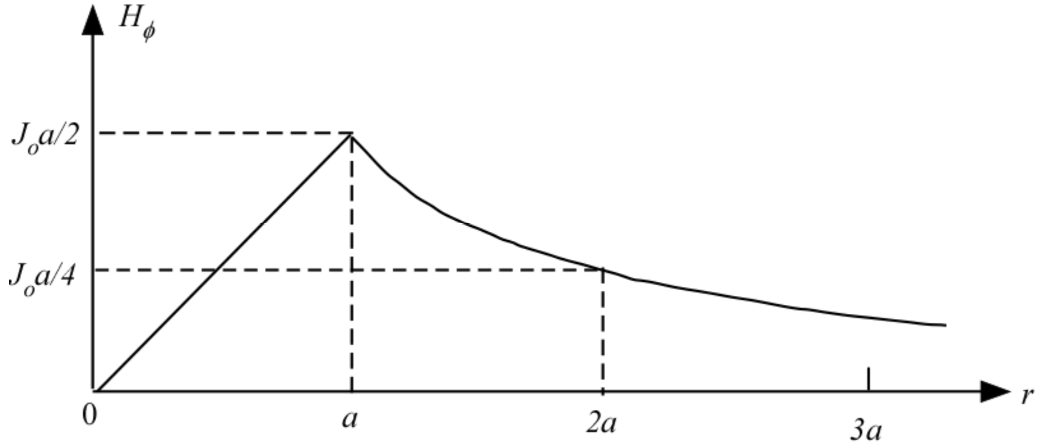
$$2\pi r H_\phi = \begin{cases} J_o \pi r^2 & for\ r \le a \\ J_o \pi a^2 & for\ r \ge a \end{cases}$$

$$H_\phi = \begin{cases} \dfrac{J_o r}{2} & for\ r \le a \\ \dfrac{J_o a^2}{2r} & for\ r \ge a \end{cases}$$



Using Gauss' Law – H Field





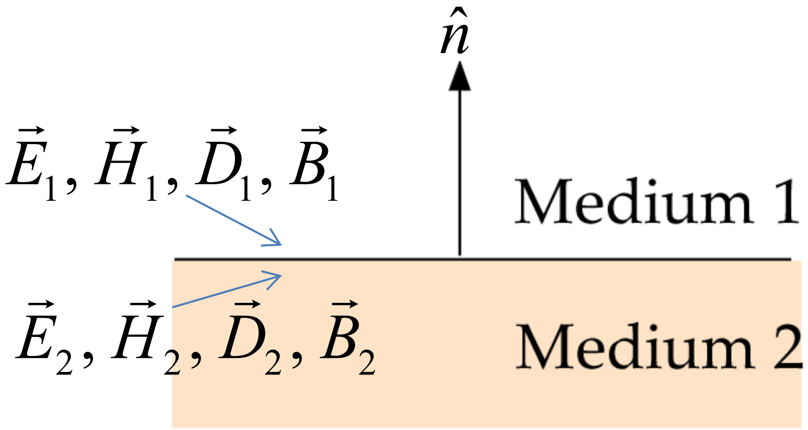
Boundary Conditions

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2 \right) = 0$$

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2 \right) = \vec{J}_S$$

$$\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2 \right) = \rho_S$$

$$\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2 \right) = 0$$



Maxwell's Equations in Free Space

$$\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t}$$

Faraday's Law of Induction

$$\nabla \times \vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Ampère's Law

$$\nabla \cdot \vec{E} = 0$$

Gauss' Law for electric field

$$\nabla \cdot \vec{B} = 0$$

Gauss' Law for magnetic field

Wave Equation

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\mu_o \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu_o \frac{\partial}{\partial t}\varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\nabla^2 \vec{E} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$ ← **Wave Equation**

can show that $$\nabla^2 \vec{H} = \mu_o \varepsilon_o \frac{\partial^2 \vec{H}}{\partial t^2}$$

Wave Equation

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

separating the components

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_z}{\partial t^2}$$

Wave Equation → Plane Wave

(a) Assume that only E_x exists → $E_y=E_z=0$

(b) Only z spatial dependence → $\frac{\partial}{\partial x}=\frac{\partial}{\partial y}=0$

$$\frac{\partial^2 E_x}{\partial z^2}=\mu_o\varepsilon_o\frac{\partial^2 E_x}{\partial t^2}$$

This situation leads to the plane wave solution

In addition, assume a time-harmonic dependence

$E_x \sim e^{j\omega t}$ then $\frac{\partial}{\partial t}\rightarrow j\omega$

Plane Wave Solution

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_o \varepsilon_o \omega^2 E_x$$

solution $\begin{cases} E_x = E_{o+} e^{-j\beta z} & \textit{forward} \quad \textit{wave} \\ E_x = E_{o-} e^{+j\beta z} & \textit{backward} \quad \textit{wave} \end{cases}$

where $\beta = \omega\sqrt{\mu_o \varepsilon_o}$ **propagation constant**

In the time domain

solution $\begin{cases} E_x(t) = E_{o+} \cos\left(\omega t - \beta z\right) & \textit{forward traveling wave} \\ E_x(t) = E_{o-} \cos\left(\omega t + \beta z\right) & \textit{backward traveling wave} \end{cases}$

Plane Wave Characteristics

where $\beta = \omega\sqrt{\mu_o \varepsilon_o}$ propagation constant

$$\frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} = v = \text{propagation velocity}$$

In free space $v = c = \dfrac{1}{\sqrt{\mu_o \varepsilon_o}} = 3 \times 10^8 \ m/s$

Solution for Magnetic Field

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \Rightarrow \nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla \times \vec{E} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

If we assume that $\vec{E} = \hat{x}E_{o+}e^{-j\beta z}$ **then** $\vec{H} = \hat{y}\frac{E_{o+}}{\eta}e^{-j\beta z}$

If we assume that $\vec{E} = \hat{x}E_{o-}e^{+j\beta z}$ **then** $\vec{H} = -\hat{y}\frac{E_{o-}}{\eta}e^{+j\beta z}$

$\eta = \sqrt{\frac{\mu}{\varepsilon}}$ **intrinsic impedance of medium**

Time-Average Poynting Vector

$$\vec{P}(t) = \vec{E}(t) \times \vec{H}(t)$$ **Poynting vector W/m²**

time-average Poynting vector W/m²

$$\langle \vec{P} \rangle = \frac{1}{T}\int_0^T \vec{P}(t)dt = \frac{1}{T}\int_0^T \vec{E}(t) \times \vec{H}(t)dt$$

We can show that

$$\langle \vec{P} \rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right]$$

where $\vec{E}$ and $\vec{H}$ are the phasors of $\vec{E}(t)$ and $\vec{H}(t)$ respectively

Material Medium

$$\nabla \times \vec{E} = -\frac{\partial \mu \vec{H}}{\partial t} \qquad \textbf{or} \qquad \nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \varepsilon \vec{E}}{\partial t} \qquad \qquad \nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E}$$

$$\vec{J} = \sigma \vec{E}$$

σ: conductivity of material medium ($\Omega^{-1}m^{-1}$)

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = \vec{E}\left(\sigma + j\omega\varepsilon\right) = j\omega\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)\vec{E}$$

since $\varepsilon \rightarrow \varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)$ **then** $\nabla^2\vec{E} = -\omega^2\mu\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)\vec{E}$

Wave in Material Medium

$$\nabla^2 \vec{E} = -\omega^2 \mu\varepsilon \left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \vec{E} = \gamma^2 \vec{E}$$

$$\gamma^2 = -\omega^2 \mu\varepsilon \left(1 + \frac{\sigma}{j\omega\varepsilon}\right)$$

γ is complex propagation constant

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{1 + \frac{\sigma}{j\omega\varepsilon}} = \alpha + j\beta$$

α: associated with attenuation of wave

β: associated with propagation of wave

Wave in Material Medium

Solution: $\vec{E} = \hat{x}E_o e^{-\gamma z} = \hat{x}E_o e^{-\alpha z} e^{-j\beta z}$ **decaying exponential**

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{1/2}$$

Complex intrinsic impedance

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Magnetic field

$$\vec{H} = \hat{y}\frac{E_o}{\eta} e^{-\alpha z} e^{-j\beta z}$$

Wave in Material Medium

Phase Velocity: $$v_p = \frac{\omega}{\beta} = \sqrt{\frac{2}{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{-1/2}$$

Wavelength: $$\lambda = \frac{2\pi}{\beta} = \frac{\sqrt{2}}{f\sqrt{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{-1/2}$$

Special Cases

1. Perfect dielectric $\sigma = 0$ air, free space

$\alpha = 0$ and $\beta = \omega\sqrt{\mu\varepsilon}$

Wave in Material Medium

2. Lossy dielectric

Loss tangent: $\dfrac{\sigma}{\omega\varepsilon} \ll 1$

$$\alpha \simeq \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left[1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right] \simeq \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\beta \simeq \omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right] \simeq \omega\sqrt{\mu\varepsilon}$$

$$\eta \simeq \sqrt{\frac{\mu}{\varepsilon}}\left[1-\frac{3\sigma^2}{8\omega^2\varepsilon^2}+j\frac{\sigma}{2\omega\varepsilon}\right]$$

Wave in Material Medium

3. Good conductors

Loss tangent: $\dfrac{\sigma}{\omega\varepsilon} \gg 1$

$$\gamma = \sqrt{j\omega\mu\left(\sigma + j\omega\varepsilon\right)} \simeq \sqrt{j\omega\mu\sigma}$$

$$\alpha = \sqrt{\pi f \mu\sigma} \qquad \beta = \sqrt{\pi f \mu\sigma}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\pi f \mu}{\sigma}}\left(1 + j\right)$$

Material Medium

	α attenuation	β propagation	η	d_p	H, E	Examples
PEC	∞	-	0	0	0	supercond
Good conductor	$\sqrt{\frac{\omega\mu\sigma}{2}}$	$\sqrt{\frac{\omega\mu\sigma}{2}}$	$\sqrt{\frac{\pi f \mu}{\sigma}}(1+j)$	$\frac{1}{\sqrt{\pi f \mu\sigma}}$	finite	copper
Poor conductor	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\omega\sqrt{\mu\varepsilon}$	$\frac{\sqrt{\mu/\varepsilon}}{\left(1-j\frac{\sigma}{2\omega\varepsilon}\right)}$	$\frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$	finite	Ice
Perfect dielectric	0	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\mu/\varepsilon}$	∞	finite	air

Radiation - Vector Potential

Assume time harmonicity ~ $e^{j\omega t}$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad (1)$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E} \qquad (2)$$

$$\nabla \cdot \vec{D} = \rho / \varepsilon \qquad (3)$$

$$\nabla \cdot \vec{B} = 0 \qquad (4)$$

Radiation - Vector Potential

Using the property: $\nabla \cdot \left(\nabla \times any\ vector \right) = 0$

$$\nabla \cdot \vec{B} = 0 \Rightarrow \exists \vec{A} \text{ such that } \nabla \times \vec{A} = \vec{B}$$

$$\nabla \cdot \left(\nabla \times \vec{A} \right) = 0$$

$\vec{A}$: vector potential

$$\nabla \times \vec{E} = -j\omega \nabla \times \vec{A} \Rightarrow \nabla \times \left(\vec{E} + j\omega \vec{A} \right) = 0$$

Vector Potential

$$\nabla \times vector = 0 \Rightarrow vector = \nabla \psi$$

$$\left(\vec{E} + j\omega\vec{A}\right) = -\nabla\phi \text{ where } \phi \text{ is the scalar potential}$$

Since a vector is uniquely defined by its curl and its divergence, we can choose the divergence of A

choose $\vec{A}$ such that

$$\nabla \cdot \vec{A} + j\omega\mu\varepsilon\phi = 0$$

Lorentz condition

Vector Potential

$$\nabla \times \vec{B} = \mu \vec{J} + j\omega\mu\varepsilon \vec{E}$$

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} + j\omega\mu\varepsilon \left(-j\omega \vec{A} - \nabla \phi \right)$$

$$-\nabla^2 \vec{A} + \nabla \left(\nabla \cdot \vec{A} \right) = \mu \vec{J} + \omega^2 \mu\varepsilon \vec{A} - j\omega\mu\varepsilon \nabla \phi$$

$$-\nabla^2 \vec{A} + \nabla \left(-j\omega\mu\varepsilon\phi \right) = \mu \vec{J} + \omega^2 \mu\varepsilon \vec{A} - j\omega\mu\varepsilon \nabla \phi$$

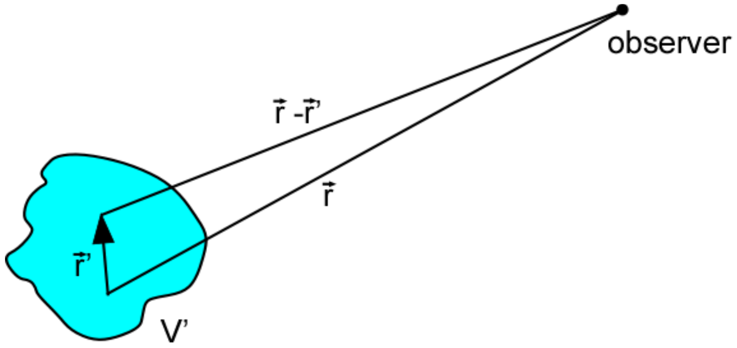
$$\nabla^2 \vec{A} + \omega^2 \mu\varepsilon \vec{A} = -\mu \vec{J}$$

D'Alembert's equation

Vector Potential

Three-dimensional free-space Green's function

$$G(\vec{r},\vec{r}\,') = \frac{e^{-j\beta|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\mu$$



Vector potential

$$\vec{A}(\vec{r}) = \mu\iiint_{V'} \frac{\vec{J}(\vec{r}\,')e^{-j\beta|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}dv'$$

From A, get E and H using Maxwell's equations

Magnetostatics

When d/dt → 0 (or ω = 0)

Poisson's **Equation** $\nabla^2 \vec{A} = -\mu \vec{J}$

The vector potential is $\vec{A}(\vec{r}) = \mu \iiint_{V'} \frac{\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} dv'$

For a thin wire carrying current I, $\mathbf{J}dV' \rightarrow I\, d\ell'$ along the wire,

$$\vec{A}(\vec{r}) = \frac{\mu I}{4\pi} \int_C \frac{d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|}$$

Inductance

The flux is
$$\Phi = \oint_C \vec{A} \cdot d\vec{l} = \frac{\mu I}{4\pi} \oint_C \oint_C \frac{d\vec{l} \cdot d\vec{l}\,'}{R}$$

where $R = \left| \vec{r} - \vec{r}\,' \right|$

Thus inductance is
$$L \triangleq \frac{\Phi}{I} = \frac{\mu}{4\pi} \oint_C \oint_C \frac{d\vec{l} \cdot d\vec{l}\,'}{R}$$