ECE 546
Lecture 03
Waveguides

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Jose E. Schutt-Aine
Electrical & Computer Engineering
University of Illinois
jesa@illinois.edu
Parallel-Plate Waveguide

Maxwell’s Equations \( \nabla^2 \mathbf{E} + \omega^2 \mu \varepsilon \mathbf{E} = 0 \)

\[
\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu \varepsilon E_x
\]

\[
\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu \varepsilon E_y
\]

\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \varepsilon E_z
\]
TE Modes

For a parallel-plate waveguide, the plates are infinite in the $y$-extent; we need to study the propagation in the $z$-direction. The following assumptions are made in the wave equation

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

$$\Rightarrow \text{Assume } E_y \text{ only}$$

These two conditions define the **TE modes** and the wave equation is simplified to read

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu \varepsilon E_y$$

(¥)
Phasor Solution

General solution (forward traveling wave)

\[ E_y(x, z) = e^{-j\beta z} \left[ Ae^{-j\beta x} + Be^{+j\beta x} \right] \]

At \( x = 0 \), \( E_y = 0 \) which leads to \( A + B = 0 \). Therefore, \( A = -B = E_o/2j \), where \( E_o \) is an arbitrary constant

\[ E_y(x, z) = E_o e^{-j\beta z} \sin \beta_x x \]

\( a \) is the distance separating the two PEC plates
Dispersion Relation

At $x = a$, $E_y(x, z) = 0 \Rightarrow E_0 e^{-j\beta z} \sin \beta_x a = 0$

This leads to: $\beta_x a = m\pi$, where $m = 1, 2, 3, ...$

$$\beta_x = \frac{m\pi}{a}$$

Moreover, from the differential equation (¥), we get the dispersion relation

$$\beta_z^2 + \beta_x^2 = \omega^2 \mu \varepsilon = \beta^2$$

which leads to

$$\beta_z = \sqrt{\omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2}$$
Guidance Condition

\[ \beta_z = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2} \]

where \( m = 1, 2, 3 \ldots \). Since propagation is to take place in the \( z \) direction, for the wave to propagate, we must have \( \beta_z^2 > 0 \), or

\[ \omega^2 \mu \varepsilon > \left( \frac{m\pi}{a} \right)^2 \]

This leads to the following guidance condition which will insure wave propagation

\[ f > \frac{m}{2a \sqrt{\mu \varepsilon}} \]
Cutoff Frequency

The cutoff frequency $f_c$ is defined to be at the onset of propagation

\[ f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} \quad \lambda_c = \frac{\nu}{f_c} = \frac{2a}{m} \]

Each mode is referred to as the TE$_m$ mode. It is obvious that there is no TE$_0$ mode and the first TE mode is the TE$_1$ mode.

The cutoff frequency is the frequency below which the mode associated with the index $m$ will not propagate in the waveguide. Different modes will have different cutoff frequencies.
Magnetic Field for TE Modes

From  \( \nabla \times \mathbf{E} = -j \omega \mu \mathbf{H} \)

we have  \( \mathbf{H} = \frac{-1}{j \omega \mu} \left| \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{array} \right| \)

which leads to

\[ H_x = -\frac{\beta_z}{\omega \mu} E_o e^{-j \beta_z z} \sin \beta_x x \]

\[ H_z = +\frac{j \beta_x}{\omega \mu} E_o e^{-j \beta_z z} \cos \beta_x x \]

The magnetic field for TE modes has 2 components
As can be seen, there is no $H_y$ component, therefore, the TE solution has $E_y$, $H_x$ and $H_z$ only.

From the dispersion relation, it can be shown that the propagation vector components satisfy the relations

$$\beta_z = \beta \sin \theta, \quad \beta_x = \beta \cos \theta$$

where $\theta$ is the angle of incidence of the propagation vector with the normal to the conductor plates.
Phase and Group Velocities

The phase and group velocities are given by

\[ v_{pz} = \frac{\omega}{\beta_z} = \frac{c}{\sqrt{1 - \frac{f_c^2}{f^2}}} \quad \text{and} \quad v_g = \frac{\partial \omega}{\partial \beta_z} = c \sqrt{1 - \frac{f_c^2}{f^2}} \]

The effective guide impedance is given by:

\[ \eta_{TE} = \frac{E_y}{-H_x} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}} \]
Transverse Magnetic (TM) Modes

The magnetic field also satisfies the wave equation:

Maxwell’s Equations ⇒ ∇²H + ω²μεH=0

\[
\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu \varepsilon H_x
\]

\[
\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu \varepsilon H_y
\]

\[
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu \varepsilon H_z
\]
**TM Modes**

For TM modes, we assume

\[ \Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0 \]

⇒ Assume \( H_y \) only

These two conditions define the *TM modes* and the equations are simplified to read

\[
\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu \varepsilon H_y
\]

**General solution (forward traveling wave)**

\[
H_y(x, z) = e^{-j\beta_z z} \left[ Ae^{-j\beta_x x} + Be^{j\beta_x x} \right]
\]
Electric Field for TM Modes

From \( \nabla \times \mathbf{H} = -j \omega \varepsilon \mathbf{E} \)

we get

\[
\mathbf{E} = \frac{1}{j \omega \varepsilon} \begin{pmatrix}
\hat{x} & \hat{y} & \hat{z}
\end{pmatrix}
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\
0 & H_y & 0
\end{bmatrix}
\]

This leads to

\[
E_x(x, z) = \frac{\beta_z}{\omega \varepsilon} e^{-j \beta_z z} \left[ Ae^{-j \beta_x x} + Be^{j \beta_x x} \right]
\]

\[
E_z(x, z) = \frac{\beta_x}{\omega \varepsilon} e^{-j \beta_z z} \left[ -Ae^{-j \beta_x x} + Be^{j \beta_x x} \right]
\]
TM Modes Fields

At $x=0$, $E_z = 0$ which leads to $A = B = H_0/2$ where $H_0$ is an arbitrary constant. This leads to

$$H_y(x,z) = H_0 e^{-j\beta_z z} \cos \beta_x x$$

$$E_x(x,z) = \frac{\beta_z}{\omega \epsilon} H_0 e^{-j\beta_z z} \cos \beta_x x$$

$$E_z(x,z) = \frac{j \beta_x}{\omega \epsilon} H_0 e^{-j\beta_z z} \sin \beta_x x$$

At $x=a$, $E_z = 0$ which leads to

$$\beta_x a = m \pi, \text{ where } m = 0, 1, 2, 3, \ldots$$
E & H Fields for TM Modes

This defines the TM modes which have only $H_y$, $E_x$ and $E_z$ components.

The effective guide impedance is given by:

$$\eta_{TM} = \frac{E_x}{H_y} = \eta_0 \sqrt{1 - \frac{f_c^2}{f^2}}$$

The electric field for TM modes has 2 components.
This defines the **TM modes**; each mode is referred to as the $\text{TM}_m$ mode. It can be seen from that $m=0$ is a valid choice; it is called the $\text{TM}_0$, or *transverse electromagnetic* or TEM mode. For this mode and,
TEM Mode

\( \beta_x = 0 \) and \( \beta_z = \beta \). There are no \( x \) variations of the fields within the waveguide. The TEM mode has a cutoff frequency at DC and is always present in the waveguide.

\[
\begin{align*}
H_y &= H_o e^{-j\beta_z z} \\
E_x &= \frac{\beta_z}{\omega \varepsilon} H_o e^{-j\beta_z z} = \sqrt{\frac{\mu}{\varepsilon}} H_o e^{-j\beta_z z} \\
E_z &= 0
\end{align*}
\]

The propagation characteristics of the TEM mode do not vary with frequency

The TEM mode is the **fundamental** mode on a parallel-plate waveguide
**Power for TE Modes**

**Time-Average Poynting Vector**  
\[ \langle P \rangle = \frac{1}{2} \text{Re}\{E \times H^*\} \]

**TE modes**

\[
\langle P \rangle = \frac{1}{2} \text{Re}\left\{ \hat{y}E_y \times \left[ \hat{x}H_x^* + \hat{z}H_z^* \right] \right\}
\]

\[
\langle P \rangle = \frac{1}{2} \text{Re}\left\{ \hat{z} \left| \frac{E_o}{\omega \mu} \right|^2 \beta_z \sin^2 \beta_x x + \hat{x}j \left| \frac{E_o}{\omega \mu} \right|^2 \beta_x \cos \beta_x x \sin \beta_x x \right\}
\]

\[
\langle P \rangle = \hat{z} \left\| \frac{E_o}{2 \omega \mu} \right\|^2 \beta_z \sin^2 \beta_x x
\]
Power for TM Modes

\[ \langle P \rangle = \frac{1}{2} \text{Re}\left\{ \left[ \hat{x}E_x + \hat{z}E_z \right] \times \hat{y}H_y^* \right\} \]

\[ \langle P \rangle = \frac{1}{2} \text{Re} \left\{ \hat{z} \frac{|H_o|^2}{\omega \varepsilon} \beta_z \cos^2 \beta_x x - \hat{x}j \frac{|H_o|^2}{\omega \varepsilon} \beta_x \sin \beta_x x \cos \beta_x x \right\} \]

\[ \langle P \rangle = \hat{z} \frac{|H_o|^2}{2 \omega \varepsilon} \beta_z \cos^2 \beta_x x \]

The total time-average power is found by integrating \( \langle P \rangle \) over the area of interest.
Waveguide

Maxwell’s Equations → \( \nabla^2 \mathbf{E} + \omega^2 \mu \varepsilon \mathbf{E} = 0 \)

\[
\begin{align*}
\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} &= -\omega^2 \mu \varepsilon E_x \\
\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} &= -\omega^2 \mu \varepsilon E_y \\
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} &= -\omega^2 \mu \varepsilon E_z
\end{align*}
\]
TE Modes

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set $E_z = 0$. This defines the TE modes.

From $\nabla \times E = -\mu \frac{\partial H}{\partial t}$, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \Rightarrow + j \beta_z E_y = - j \omega \mu H_x \quad (1)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \Rightarrow - j \beta_z E_x = - j \omega \mu H_y \quad (2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = - j \omega \mu H_z \quad (3)$$
TE Modes

From \( \nabla \times \mathbf{H} = j \omega \varepsilon \mathbf{E} \), we get

\[
\begin{align*}
    j \omega \varepsilon \mathbf{E} = & \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \\
    \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j \omega \varepsilon E_x & \Rightarrow \frac{\partial H_z}{\partial y} + j \beta_z H_y = j \omega \varepsilon E_x \quad (4) \\
    \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y & \Rightarrow -j \beta_z H_x - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y \quad (5) \\
    \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 & \quad (6)
\end{align*}
\]

We want to express all quantities in terms of \( H_z \).
TE Modes

From (2), we have \( H_y = \frac{\beta_z E_x}{\omega \mu} \)

in (4) \( \frac{\partial H_z}{\partial y} + j \beta_z^2 E_x = j \omega \epsilon E_x \)

Solving for \( E_x \)
\( E_x = \frac{j \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} \frac{\partial H_z}{\partial y} \)

From (1) \( H_x = \frac{-\beta_z E_y}{\omega \mu} \)

in (5) \( j \frac{\beta_z^2 E_y}{\omega \mu} - \frac{\partial H_z}{\partial x} = j \omega \epsilon E_y \)

so that \( E_y = \frac{-j \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} \frac{\partial H_z}{\partial x} \)
TE Modes

\[ H_y = \frac{j \beta_z}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial y} \]

\[ H_x = \frac{j \beta_z}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial x} \]

\[ E_z = 0 \]

Combining solutions for \( E_x \) and \( E_y \) into (3) gives

\[ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] H_z \quad (\text{Y}) \]
Rectangular Waveguide

If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

\[ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] H_z \]  

(𝑌)
**TE Modes**

The general solution for TE modes with $E_z = 0$ is obtained from (Y)

$$H_z = e^{-j\beta_z z} \left[ A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[ C e^{-j\beta_y y} + D e^{+j\beta_y y} \right]$$

$$E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_z z} \left[ -A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[ C e^{-j\beta_y y} + D e^{+j\beta_y y} \right]$$

$$E_x = \frac{-\beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_z z} \left[ A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[ -C e^{-j\beta_y y} + D e^{+j\beta_y y} \right]$$

At $y=0$, $E_x=0$ which leads to $C=D$

At $x=0$, $E_y=0$ which leads to $A=B$
TE Modes

\[ H_z = H_0 e^{-j\beta_z z} \cos \beta_x x \cos \beta_y y \]  
\[ E_y = \frac{j \beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z} \sin \beta_x x \cos \beta_y y \]  
\[ E_x = \frac{-j \beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z} \cos \beta_x x \sin \beta_y y \]  

At \( x=a \), \( E_y=0 \) which leads to \( \beta_x = \frac{m\pi}{a} \)  

At \( y=b \), \( E_x=0 \) which leads to \( \beta_y = \frac{n\pi}{b} \)  

The general solution for TE modes with \( E_z=0 \) is
Dispersion Relation

The dispersion relation is obtained by placing (§) in (¥)

\[ \beta_z^2 + \beta_x^2 + \beta_y^2 = \omega^2 \mu \varepsilon \quad (23) \]

\[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \beta_z^2 = \omega^2 \mu \varepsilon \quad (24) \]

\[ \beta_z = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \quad (25) \]

The guidance condition is

\[ \omega^2 \mu \varepsilon > \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \quad (26) \]
Guidance Condition

or $f > f_c$ where $f_c$ is the cutoff frequency of the TE$_{mn}$ mode given by the relation

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The TE$_{mn}$ mode will not propagate unless $f$ is greater than $f_c$.

Obviously, different modes will have different cutoff frequencies.
The transverse magnetic modes for a general waveguide are obtained by assuming $H_z = 0$. By duality with the TE modes, we have

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] E_z$$

$$E_z = e^{-j\beta_z z} \left[ Ae^{-j\beta_x x} + Be^{+j\beta_x x} \right] \left[ Ce^{-j\beta_y y} + De^{+j\beta_y y} \right]$$
TM Mode

The boundary conditions are

At $x=0, \ E_z=0$ which leads to $A=-B$

At $y=0, \ E_z=0$ which leads to $C=-D$

At $x=a, \ E_z=0$ which leads to $\beta_x = \frac{m\pi}{a}$

At $y=b, \ E_z=0$ which leads to $\beta_y = \frac{n\pi}{b}$
TM and TE Modes

so that the generating equation for the TM\(_{mn}\) modes is

\[ E_z = E_0 e^{-j\beta_z z} \sin \beta_x x \sin \beta_y y \]

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

For additional information on the field equations see Rao (6\(^{th}\) Edition), page 607, Table 9.1.
TE and TM Modes

There is no TE_{00} mode

There are no TM_{m0} or TM_{0n} modes

The first TE mode is the TE_{10} mode

The first TM mode is the TM_{11} mode
Impedance of a Waveguide

For a TE mode, we define the transverse impedance as

\[ \eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega \mu}{\beta_z} \]

From the relationship for \( \beta_z \) and using

we get

\[ f_c^2 = \frac{1}{4 \mu \varepsilon} \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right] \]

\[ \eta_{gTE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}} \]

where \( \eta \) is the intrinsic impedance

\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} \]
Impedance of a Waveguide

Analogously, for TM modes, it can be shown that

$$\eta_{gTM} = \eta \sqrt{1 - \frac{f_c^2}{f^2}}$$
Power Flow in a Waveguide

**TE_{10} Mode**

The time-average Poynting vector for the TE_{10} mode in a rectangular waveguide is given by

\[
\langle P \rangle = \frac{1}{2} \text{Re} \left[ E \times H^* \right] = \hat{z} \frac{|E_o|^2}{2} \frac{\beta_z}{\omega \mu} \sin^2 \frac{\pi x}{a}
\]

\[
\langle Power \rangle = \int_0^a \int_0^b \frac{|E_o|^2}{2} \frac{\beta_z}{\omega \mu} \sin^2 \frac{\pi x}{a} \, dx \, dy
\]

\[
\langle Power \rangle = \frac{|E_o|^2}{4} \frac{\beta_z ab}{\omega \mu} = \frac{|E_o|^2}{4} \frac{ab}{\eta_{gTE_{10}}}
\]

The time-average power flow in a waveguide is proportional to its cross-section area.
Circular Waveguide - Fields

For a waveguide with arbitrary cross section, it is known that

**TE Modes** \[ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] H_z \] \quad (1)

**TM Modes** \[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] E_z \] \quad (2)

We first assume TM modes in cylindrical coordinates:

\[ \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + \left( \gamma^2 + \omega^2 \mu \varepsilon \right) E_z = 0 \]

\[ \nabla^2_{tr} E_z \]

\[ \gamma = \pm j \beta_z \]

See Reference [6].
Circular Waveguide – TM Modes

Solution will be in the form

\[ E_z(r, \phi) = f(r)g(\phi) \]

Which after substitution gives

\[ \frac{r}{f} \frac{d}{dr}\left( r \frac{df}{dr} \right) + h^2 r^2 = -\frac{1}{g} \frac{d^2 g}{d\phi^2} \quad (3) \]

where \( h^2 = \gamma^2 + \omega^2 \mu \varepsilon \)

For equality in (3) to hold, both sides must be equal to the same constant say \( n^2 \) where \( n \) is an integer in view of the azimuthal symmetry since the fields must be periodic in \( \phi \).
Circular Waveguide – TM Modes

\[
\frac{d^2 g}{d\phi^2} + n^2 g = 0 \quad (4)
\]

\[
\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} \left( h^2 - \frac{n^2}{r^2} \right) f = 0 \quad (5)
\]

Solution of (4) is of the form

\[
g(\phi) = C_1 \cos(n\phi) + C_2 \sin(n\phi) \quad (6)
\]

(5) is Bessel’s equation and has solution

\[
f(r) = C_3 J_n(hr) + C_4 Y_n(hr) \quad (7)
\]

\( J_n \) and \( Y_n \) are the \( n^{\text{th}} \) order Bessel functions of the first and second kinds respectively
Bessel Functions of the First Kind

\[ J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r! \Gamma(n + r + 1)} \]

\[ \Gamma(n + 1) = n! \]
Circular Waveguide – TM Modes

$Y_n$ has singularity at 0 and must consequently be discarded $\Rightarrow C_4 = 0$. The general solution then becomes

$$E_z(r,\phi) = C_3 J_n(hr)[C_1 \cos(n\phi) + C_2 \sin(n\phi)]$$

Since the origin for $\phi$ is arbitrary, the expression can be written as:

$$E_z(r,\phi) = C_n J_n(hr) \cos(n\phi)$$

where $C_n$ is a constant. The boundary condition $E_{tan} = 0$ requires that

$$E_z(r,\phi) = 0 \text{ for } r = a$$

Solution exists for only discrete values of $h$ such that

$$J_n(ha) = 0$$
Circular Waveguide – TM Modes

ha must be a root of the $n^{th}$ order Bessel function. If we assume that $t_{nl}$ is the $l^{th}$ root of $J_n$, we can define a set of eigenvalues $h_{nl}$ for the TM modes so that:

$$h_{TM_{nl}} = \frac{t_{nl}}{a}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
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<td>1</td>
<td>2.405</td>
<td>3.832</td>
<td>5.136</td>
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<td>2</td>
<td>5.520</td>
<td>7.016</td>
<td>8.417</td>
</tr>
<tr>
<td>3</td>
<td>8.654</td>
<td>13.323</td>
<td>11.620</td>
</tr>
</tbody>
</table>

Each choice of $n$ and $l$ specifies a particular solution or mode.

$n$ is related to the number of circumferential variations and $l$ describes the number of radial variations of the field.
Circular Waveguide – TM Modes

The propagation constant of the $nl^{th}$ propagating TM mode is:

$$\beta_{TM_{nl}} = \left[ \omega^2 \mu \varepsilon - \left( \frac{t_{nl}}{a} \right)^2 \right]^{1/2}$$

The propagation occurs for $\lambda < \lambda_{cTM_{nl}}$ or $f > f_{cTM_{nl}}$ where the cutoff frequency and wavelength can be found from $\gamma = 0$ as:

$$\lambda_{cTM_{nl}} = \frac{2\pi a}{t_{nl}} \quad f_{cTM_{nl}} = \frac{t_{nl}}{2\pi a \sqrt{\mu \varepsilon}}$$

The other field components can be obtained from $E_z$

$$E_z = C_n J_n \left( \frac{t_{nl}}{a} r \right) \cos(n\phi) e^{-j\beta_{nl} z}$$
Circular Waveguide – TE Modes

The solutions for the TE modes can be found in a similar manner except that we solve for $H_z(r, \phi)$ to get:

$$H_z(r, \phi) = C_n J_n(hr) \cos(n\phi)$$

To apply the boundary condition $E_{tan} = 0$, we require

$$\frac{\partial H_z}{\partial r} \text{ to be 0 at } r = a$$

We must have

$$\hat{n} \cdot \nabla_{tr} H_z = \frac{\partial H_z}{\partial r} = 0 \text{ at } r = a$$

For this, we need the zeros of $J_n'(u)$ given by $s_{nl}$. The propagation constant, cutoff frequency and wavelength have the same expressions as in the TM case with $t_{nl} \rightarrow s_{nl}$. 
Circular Waveguide – TE Modes

The propagation constant of the $nl^{th}$ propagating TE mode is:

$$\beta_{TE_{nl}} = \left[ \omega^2 \mu \varepsilon - \left( \frac{S_{nl}}{a} \right)^2 \right]^{1/2}$$

$l^{th}$ root of $J_n'(.) = 0$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.832</td>
<td>1.841</td>
<td>3.054</td>
</tr>
<tr>
<td>2</td>
<td>7.016</td>
<td>5.331</td>
<td>6.706</td>
</tr>
<tr>
<td>3</td>
<td>10.173</td>
<td>8.536</td>
<td>9.969</td>
</tr>
</tbody>
</table>

From the tables, it can be seen that the lowest cutoff frequency is the TE$_{11}$ mode.

and for TE modes,

$$H_z = C_n J_n \left( \frac{S_{nl}}{a} r \right) \cos(n\phi) e^{-j\beta_{nl}z}$$
Circular Waveguide – TE & TM Modes

See Reference [6].
$\text{TE}_{11}$ Mode in Circular Waveguide

See Reference [1].
Modes in Circular Waveguide

See Reference [1].
Example: Circular Waveguide Design

Design an air-filled circular waveguide such that only the dominant mode will propagate over a bandwidth of 10 GHz.

Solution: the cutoff frequency of the TE_{11} mode is the lower bound of the bandwidth.

\[ f_{c_{TE_{11}}} = \frac{1.8412c}{2\pi a} \]

The next mode is the TM_{01} with cutoff frequency:

\[ f_{c_{TM_{01}}} = \frac{2.4049c}{2\pi a} \]
Example: Circular Waveguide Design

The BW is the difference between these two frequencies

\[ BW = f_{cTM_{01}} - f_{cTE_{11}} = \frac{c}{2\pi a} (2.4049 - 1.8412) = 10GHz \]

From which we find \( a = 0.269 \) cm

So that

\[ f_{cTE_{11}} = 32.7 \text{ GHz and } f_{cTM_{11}} = 42.76 \text{ GHz} \]
Coaxial Waveguide

- Most common two-conductor transmission system
- Dielectric filling in most microwave applications is polyethylene or Teflon
Coaxial Waveguide – TEM Mode

- Two-conductor system → Dominant mode is TEM
- Tangential E-field and normal H field must be 0 in conductor surfaces

\[ E_\phi = 0 \text{ and } H_r = 0 \text{ at } r = a, b \]
Coaxial Waveguide – TEM Mode

TEM solution can exist only with

\[ E = \hat{r}E_r(r,z) \quad \text{and} \quad H = \hat{\phi}H_\phi(r,z) \]

with no \( \phi \) dependence because of azimuthal symmetry.

we get

\[-\frac{\partial H_\phi}{\partial z} = j\omega E_r \rightarrow j\beta H_\phi^0(r) = j\omega \varepsilon E_r^0(r)\]

\[-\frac{1}{r} H_\phi + \frac{\partial H_\phi}{\partial r} = 0 \rightarrow -\frac{1}{r} H_\phi^0(r) + \frac{\partial H_\phi^0}{\partial r} = 0\]

Where propagation in \( z \) direction is assumed.
Coaxial Waveguide – TEM Mode

We get

\[
H = \hat{\phi} \frac{H_o}{r} e^{-j\beta z} \quad \quad \quad \quad \quad \quad \quad \quad \quad E = \hat{r} \frac{H_o \eta}{r} e^{-j\beta z}
\]

where \( H_o \) is a constant. No cutoff condition for TEM mode.

The voltage between the two conductors is given by

\[
V(z) = -\eta H_o \ln (b / a) e^{-j\beta z}
\]

The current in the inner conductor is given by

\[
I(z) = 2\pi H_o e^{-j\beta z}
\]

The characteristic impedance \( Z_o \) is thus given by

\[
Z_o = \eta \frac{\ln(b / a)}{2\pi}
\]
Coaxial Waveguide – TE and TM Modes

TE and TM modes may also exist in addition to TEM. In a coaxial line, they are generally undesirable.

For TM modes, we have:

\[ E_z^\circ (r, \phi) = \left[ C_3 J_n (hr) + C_4 Y_n (hr) \right] \cos(n\phi) \]

For TE modes, we have:

\[ H_z^\circ (r, \phi) = \left[ C_3' J_n (hr) + C_4' Y_n (hr) \right] \cos(n\phi) \]

With boundary conditions at \( r = a, b \) of

\[ E_z (r, \phi) = 0 \quad \text{for TM modes} \]

\[ \frac{\partial H_z}{\partial r} = 0 \quad \text{for TE modes} \]
Coaxial Waveguide – TE and TM Modes

These conditions lead to

\[ J_n(ha)Y_n(hb) = J_n(hb)Y_n(ha) \quad \text{for TM modes} \]

\[ J'_n(ha)Y'_n(hb) = J'_n(hb)Y'_n(ha) \quad \text{for TE modes} \]

Solutions of these transcendental equations determine the eigenvalues of \( h \) for given \( a, b \). As in the circular waveguide case, the modes for coaxial waveguide are denoted \( \text{TE}_{nl} \) and \( \text{TM}_{nl} \).
Coaxial Waveguide – TE and TM Modes

The mode with the lowest cutoff frequency is the $\text{TE}_{11}$ mode for which the eigenvalue $h$ is approximated as:

$$h = \frac{2}{a + b}$$

The cutoff frequency and cutoff wavelength are given by

$$\lambda_{c11} = \frac{2\pi}{h} \approx \pi(a + b) \quad \text{and} \quad f_{c11} \approx \frac{1}{\pi(a + b)\sqrt{\mu\varepsilon}}$$
Coaxial Waveguide – TE and TM Modes

See Reference [3].
References


