ECE 546
Lecture 03
Waveguides

Spring 2026

Jose E. Schutt-Aine
Electrical & Computer Engineering
University of Illinois
jesa@illinois.edu

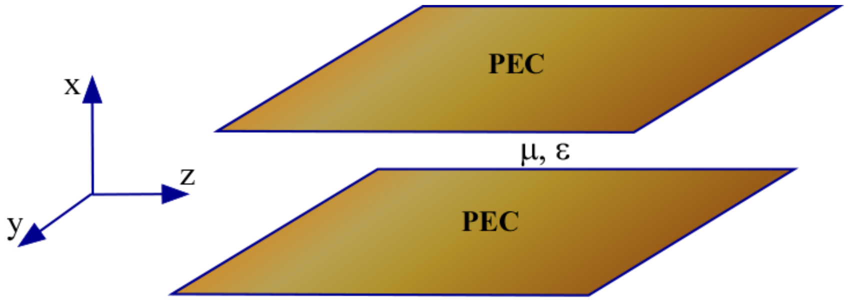
Parallel-Plate Waveguide

Maxwell's Equations ➔ $\nabla^2 \mathbf{E} + \omega^2 \mu\varepsilon \mathbf{E} = \mathbf{0}$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu\varepsilon E_x$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu\varepsilon E_y$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu\varepsilon E_z$$



TE Modes

For a parallel-plate waveguide, the plates are infinite in the y-extent; we need to study the propagation in the z-direction. The following assumptions are made in the wave equation

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

$$\Rightarrow \text{Assume } E_y \text{ only}$$

These two conditions define the **TE modes** and the wave equation is simplified to read

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu\varepsilon E_y \qquad (¥)$$

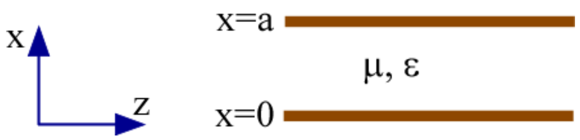
Phasor Solution

General solution (forward traveling wave)

$$E_y(x,z) = e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x} \right]$$

At $x = 0$, $E_y = 0$ which leads to $A + B = 0$. Therefore, $A = -B = E_o/2j$, where E_o is an arbitrary constant

$$E_y(x,z) = E_o e^{-j\beta_z z} \sin \beta_x x$$



a is the distance separating the two PEC plates

Dispersion Relation

At $x = a$, $E_y(x, z) = 0$ ➔ $E_o e^{-j\beta_z z} \sin \beta_x a = 0$

This leads to: $\beta_x a = m\pi$, where $m = 1, 2, 3, \ldots$

$$\beta_x = \frac{m\pi}{a}$$

Moreover, from the differential equation (¥), we get the *dispersion relation*

$$\beta_z^2 + \beta_x^2 = \omega^2 \mu\varepsilon = \beta^2$$

which leads to $\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$

Guidance Condition

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

where m = 1, 2, 3 ... Since propagation is to take place in the z direction, for the wave to propagate, we must have $\beta_z^2 > 0$, or

$$\omega^2 \mu\varepsilon > \left(\frac{m\pi}{a}\right)^2$$

This leads to the following *guidance condition* which will insure wave propagation

$$f > \frac{m}{2a\sqrt{\mu\varepsilon}}$$

Cutoff Frequency

The *cutoff frequency* f_c is defined to be at the onset of propagation

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} \qquad \lambda_c = \frac{v}{f_c} = \frac{2a}{m}$$

Each mode is referred to as the TE_m mode. It is obvious that there is no TE_0 mode and the first TE mode is the TE_1 mode.

The *cutoff frequency* is the frequency below which the mode associated with the index ***m*** will not propagate in the waveguide. Different modes will have different cutoff frequencies.

Magnetic Field for TE Modes

From $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

we have $\mathbf{H} = \dfrac{-1}{j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$

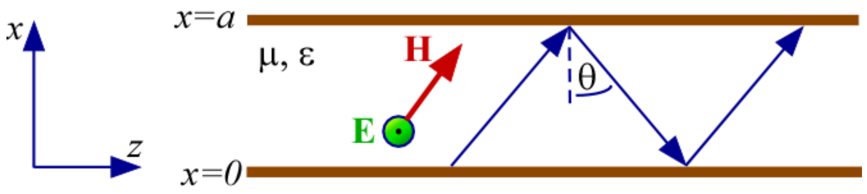
which leads to

$$H_x = -\frac{\beta_z}{\omega\mu} E_o e^{-j\beta_z z} \sin \beta_x x$$

$$H_z = +\frac{j\beta_x}{\omega\mu} E_o e^{-j\beta_z z} \cos \beta_x x$$

The magnetic field for TE modes has 2 components

E & H Fields for TE Modes



As can be seen, there is no H_y component, therefore, the TE solution has E_y, H_x and H_z only.

From the dispersion relation, it can be shown that the propagation vector components satisfy the relations
$\beta_z = \beta \sin\theta$, $\beta_x = \beta \cos\theta$ where θ is the angle of incidence of the propagation vector with the normal to the conductor plates.

Phase and Group Velocities

The phase and group velocities are given by

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{c}{\sqrt{1 - \frac{f_c^2}{f^2}}} \quad \text{and} \quad v_g = \frac{\partial \omega}{\partial \beta_z} = c\sqrt{1 - \frac{f_c^2}{f^2}}$$

The effective guide impedance is given by:

$$\eta_{TE} = \frac{E_y}{-H_x} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

Transverse Magnetic (TM) Modes

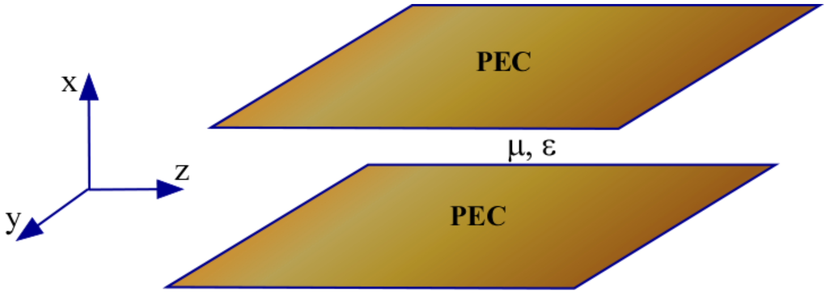
The magnetic field also satisfies the wave equation:

Maxwell's Equations ➔ $\nabla^2 \mathbf{H} + \omega^2 \mu\varepsilon \mathbf{H} = 0$

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} = -\omega^2 \mu\varepsilon H_x$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu\varepsilon H_y$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu\varepsilon H_z$$



TM Modes

For TM modes, we assume

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

➔Assume H_y only

These two conditions define the *TM modes* and the equations are simplified to read

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu \varepsilon H_y$$

<u>General solution (forward traveling wave)</u>

$$H_y(x,z) = e^{-j\beta_z z}\left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right]$$

Electric Field for TM Modes

From $\nabla \times \mathbf{H} = -j\omega\varepsilon\mathbf{E}$

we get $$\mathbf{E} = \frac{1}{j\omega\varepsilon}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix}$$

This leads to

$$E_x(x,z) = \frac{\beta_z}{\omega\varepsilon} e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]$$

$$E_z(x,z) = \frac{\beta_x}{\omega\varepsilon} e^{-j\beta_z z}\left[-Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]$$

TM Modes Fields

At $x=0$, $E_z = 0$ which leads to $A = B = H_o/2$ where H_o is an arbitrary constant. This leads to

$$H_y(x,z) = H_o e^{-j\beta_z z} \cos \beta_x x$$

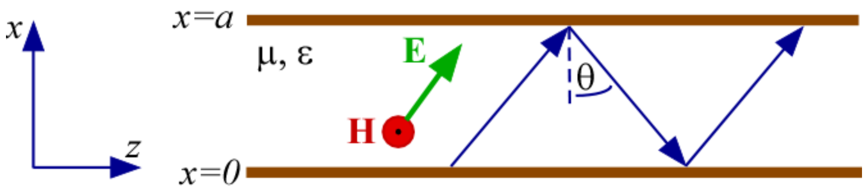
$$E_x(x,z) = \frac{\beta_z}{\omega\varepsilon} H_o e^{-j\beta_z z} \cos \beta_x x$$

$$E_z(x,z) = \frac{j\beta_x}{\omega\varepsilon} H_o e^{-j\beta_z z} \sin \beta_x x$$

At $x = a$, $E_z = 0$ which leads to

$\beta_x a = m\pi$, where $m = 0, 1, 2, 3, \ldots$

E & H Fields for TM Modes



$$\beta_x = \frac{m\pi}{a}$$

This defines the TM modes which have only H_y, E_x and E_z components.

The effective guide impedance is given by:

$$\eta_{TM} = \frac{E_x}{H_y} = \eta_o \sqrt{1 - \frac{f_c^2}{f^2}}$$

The electric field for TM modes has 2 components

E & H Fields for TM Modes

THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A PARALLEL-PLATE WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

This defines the **TM modes**; each mode is referred to as the TM_m mode. It can be seen from that $m=0$ is a valid choice; it is called the TM_0, or *transverse electromagnetic* or TEM mode. For this mode and,

TEM Mode

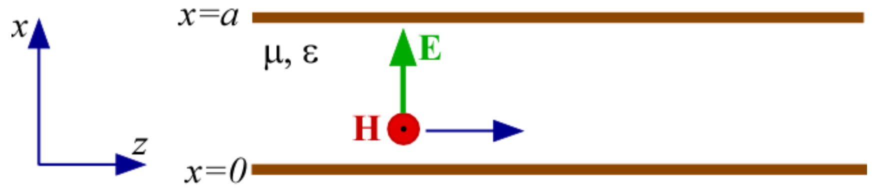
β_x=0 and $\beta_z = \beta$. There are no x variations of the fields within the waveguide. The TEM mode has a cutoff frequency at DC and is always present in the waveguide.

$$H_y = H_o e^{-j\beta_z z}$$

$$E_x = \frac{\beta_z}{\omega\varepsilon} H_o e^{-j\beta_z z} = \sqrt{\frac{\mu}{\varepsilon}} H_o e^{-j\beta_z z}$$

$$E_z = 0$$

The propagation characteristics of the TEM mode do not vary with frequency



The TEM mode is the *fundamental* mode on a parallel-plate waveguide

Power for TE Modes

Time-Average Poynting Vector $\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \mathbf{E} \times \mathbf{H}^* \right\}$

<u>TE modes</u>

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \hat{\mathbf{y}} E_y \times \left[\hat{\mathbf{x}} H_x^* + \hat{\mathbf{z}} H_z^* \right] \right\}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \hat{\mathbf{z}} \frac{|E_o|^2}{\omega\mu} \beta_z \sin^2 \beta_x x + \hat{\mathbf{x}} j \frac{|E_o|^2}{\omega\mu} \beta_x \cos \beta_x x \sin \beta_x x \right\}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}} \frac{|E_o|^2}{2\omega\mu} \beta_z \sin^2 \beta_x x$$

Power for TM Modes

TM modes

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \left[\hat{\mathbf{x}}E_x + \hat{\mathbf{z}}E_z \right] \times \hat{\mathbf{y}}H_y^* \right\}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \hat{\mathbf{z}}\frac{|H_o|^2}{\omega\varepsilon}\beta_z \cos^2\beta_x x - \hat{\mathbf{x}}j\frac{|H_o|^2}{\omega\varepsilon}\beta_x \sin\beta_x x \cos\beta_x x \right\}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}}\frac{|H_o|^2}{2\omega\varepsilon}\beta_z \cos^2\beta_x x$$

The total time-average power is found by integrating <**P**> over the area of interest.

Example 1

Consider an air-filled parallel-plate waveguide and a frequency of operation of 2.5 GHz.

(a) Determine the maximum distance between the plates that allow propagation of the fundamental mode only.

(b) The waveguide is now filled with a dielectric of ε_r=10. Find all the propagating modes at 2.5 GHz

$$f_{cTE_1} = \frac{c}{2a} \Rightarrow a = \frac{c}{2 f_{cTE_1}}$$

$$a = \frac{0.3\times 10^9}{2\times 2.5\times 10^9} = \frac{0.3}{5} = 0.06 \text{ m} = 6 \text{ cm}$$

Example 1

$$f_{cTE_1} = \frac{v}{2a} = \frac{c}{\sqrt{10}} \cdot \frac{1}{2\times 6\times 10^{-2}} = \frac{2.5}{\sqrt{10}} = 0.7905\ GHz$$

$$f_{cTE_2} = 2\times 0.7905\ \text{GHz} = 1.58\ \text{GHz}$$

$$f_{cTE_3} = 3\times 0.7905\ \text{GHz} = 2.37\ \text{GHz}$$

Modes propagating at 2.5 GHz

MODE	Cutoff Frequency
TEM	DC
TE_1	0.79 GHz
TM_1	0.79 GHz
TE_2	1.58 GHz
TM_2	1.58 GHz
TE_3	2.37 GHz
TM_3	2.37 GHz

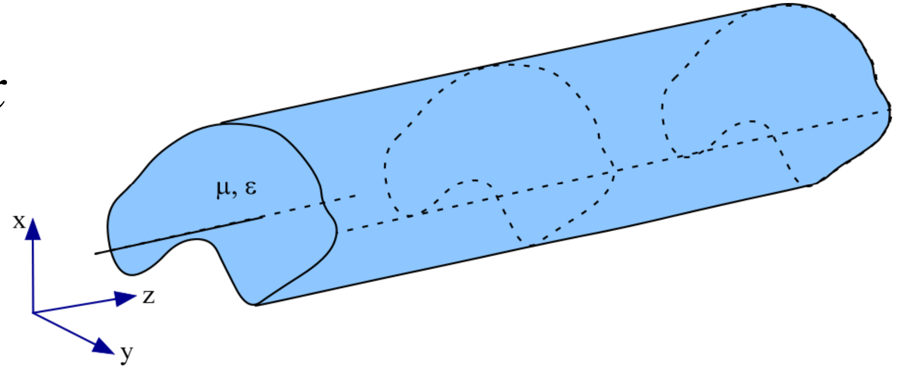
Waveguide

Maxwell's Equations ➔ $\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = \mathbf{0}$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2\mu\varepsilon E_x$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2\mu\varepsilon E_y$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2\mu\varepsilon E_z$$



TE Modes

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set $E_z = 0$. This defines the TE modes.

From $\nabla \times \mathbf{E} = -\mu \dfrac{\partial \mathbf{H}}{\partial t}$, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \Rightarrow +j\beta_z E_y = -j\omega\mu H_x \quad (1)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \Rightarrow -j\beta_z E_x = -j\omega\mu H_y \quad (2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (3)$$

TE Modes

From $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$, we get $\quad j\omega\varepsilon\mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega\varepsilon E_x \quad (4)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \Rightarrow -j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (6)$$

We want to express all quantities in terms of *H_z*.

TE Modes

From (2), we have $H_y = \dfrac{\beta_z E_x}{\omega\mu}$

in (4) $\dfrac{\partial H_z}{\partial y} + j\beta_z^2 \dfrac{E_x}{\omega\mu} = j\omega\varepsilon E_x$

Solving for E_x $\quad E_x = \dfrac{j\omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon}\dfrac{\partial H_z}{\partial y}$

From (1) $\quad H_x = \dfrac{-\beta_z E_y}{\omega\mu}$

in (5) $j\dfrac{\beta_z^2 E_y}{\omega\mu} - \dfrac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$

so that $\quad E_y = \dfrac{-j\omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon}\dfrac{\partial H_z}{\partial x}$

TE Modes

$$H_y = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial y}$$

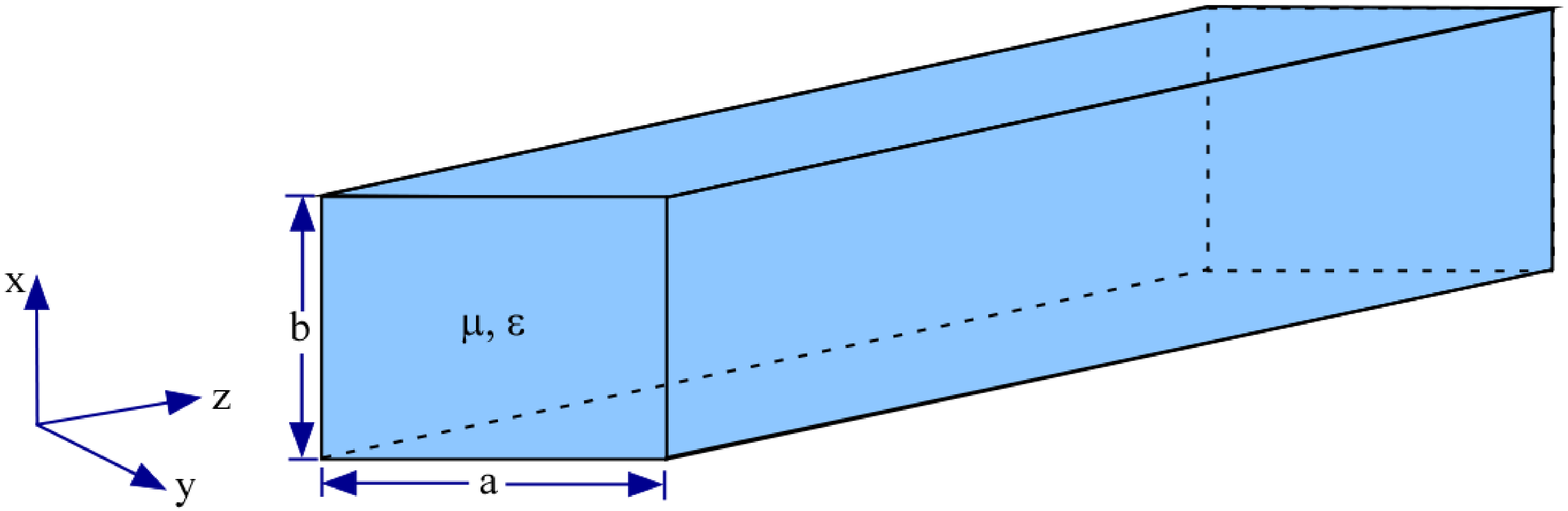
$$H_x = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial x}$$

$$E_z = 0$$

Combining solutions for E_x and E_y into (3) gives

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[\beta_z^2 - \omega^2 \mu\varepsilon\right] H_z \qquad (¥)$$

Rectangular Waveguide



$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[\beta_z^2 - \omega^2 \mu\varepsilon\right] H_z \qquad (¥)$$

If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

TE Modes

The general solution for TE modes with $E_z=0$ is obtained from (¥)

$$H_z = e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y} + De^{+j\beta_y y}\right]$$

$$E_y = \frac{\beta_x \omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon} e^{-j\beta_z z}\left[-Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y} + De^{+j\beta_y y}\right]$$

$$E_x = \frac{-\beta_y \omega\mu}{\beta_z^2 - \omega^2\mu\varepsilon} e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right]\left[-Ce^{-j\beta_y y} + De^{+j\beta_y y}\right]$$

At $y = 0$, $E_x = 0$ which leads to $C = D$

At $x = 0$, $E_y = 0$ which leads to $A = B$

TE Modes

$$H_z = H_o e^{-j\beta_z z} \cos\beta_x x \cos\beta_y y \qquad (\S)$$

$$E_y = \frac{j\beta_x \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} H_o e^{-j\beta_z z} \sin\beta_x x \cos\beta_y y$$

$$E_x = \frac{-j\beta_y \omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} H_o e^{-j\beta_z z} \cos\beta_x x \sin\beta_y y$$

At x = a, $E_y = 0$ which leads to $\beta_x = \dfrac{m\pi}{a}$

At y = b, $E_x = 0$ which leads to $\beta_y = \dfrac{n\pi}{b}$

The general solution for TE modes with $E_z=0$ is

Dispersion Relation

The dispersion relation is obtained by placing (§) in (¥)

$$\beta_z^2 + \beta_x^2 + \beta_z^2 = \omega^2 \mu\varepsilon \qquad (23)$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \omega^2 \mu\varepsilon \qquad (24)$$

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad (25)$$

The guidance condition is

$$\omega^2 \mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad (26)$$

Guidance Condition

or $f > f_c$ where f_c is the cutoff frequency of the TE_{mn} mode given by the relation

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The TE_{mn} mode will not propagate unless f is greater than f_c.

Obviously, different modes will have different cutoff frequencies.

TM Mode

The transverse magnetic modes for a general waveguide are obtained by assuming H_z =0. By duality with the TE modes, we have

$$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}=\left[\beta_z^2-\omega^2\mu\varepsilon\right]E_z$$

$$E_z=e^{-j\beta_z z}\left[Ae^{-j\beta_x x}+Be^{+j\beta_x x}\right]\left[Ce^{-j\beta_y y}+De^{+j\beta_y y}\right]$$

TM Mode

The boundary conditions are

At $x = 0$, $E_z = 0$ which leads to $A = -B$

At $y = 0$, $E_z = 0$ which leads to $C = -D$

At $x = a$, $E_z = 0$ which leads to $\beta_x = \dfrac{m\pi}{a}$

At $y = b$, $E_z = 0$ which leads to $\beta_y = \dfrac{n\pi}{b}$

TM and TE Modes

so that the generating equation for the TM_{mn} modes is

$$E_z = E_o e^{-j\beta_z z} \sin \beta_x x \sin \beta_y y$$

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

For additional information on the field equations see **Rao (6th Edition), page 607, Table 9.1**.

TE and TM Modes

There is no TE_{00} mode

There are no TM_{m0} or TM_{0n} modes

The first TE mode is the TE_{10} mode

The first TM mode is the TM_{11} mode

Impedance of a Waveguide

For a TE mode, we define the transverse impedance as

$$\eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta_z}$$

From the relationship for β_z and using

we get $$f_c^2 = \frac{1}{4\mu\varepsilon}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$$

$$\eta_{gTE} = \frac{\eta}{\sqrt{1-\frac{f_c^2}{f^2}}}$$ where η is th intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

Impedance of a Waveguide

Analogously, for TM modes, it can be shown that

$$\eta_{gTM} = \eta\sqrt{1-\frac{f_c^2}{f^2}}$$

Power Flow in a Waveguide

TE_{10} Mode

The time-average Poynting vector for the TE_{10} mode in a rectangular waveguide is given by

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right] = \hat{\mathbf{z}}\frac{|E_o|^2}{2}\frac{\beta_z}{\omega\mu}\sin^2\frac{\pi x}{a}$$

$$\langle Power \rangle = \int_0^a \int_0^b \frac{|E_o|^2}{2}\frac{\beta_z}{\omega\mu}\sin^2\frac{\pi x}{a}dxdy$$

$$\langle Power \rangle = \frac{|E_o|^2}{4}\frac{\beta_z ab}{\omega\mu} = \frac{|E_o|^2}{4}\frac{ab}{\eta_{gTE_{10}}}$$

The time-average power flow in a waveguide is proportional to its cross-section area.

Problem 2

A 10-meter section of air-filled rectangular waveguide has dimensions 2.5 cm x 1 cm.

(a) Find all the modes propagating below 18 GHz and their respective cutoff frequencies.

(b) For TE_{10} mode operation, what is the time delay difference between a 10 GHz pulse and a 7 GHz pulse?

(a)
$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$TE_{10} \rightarrow \frac{c}{2a} = \frac{0.3\times 10^9}{2\times 0.025} = 6 \text{ GHz}$$

$$TE_{20} \rightarrow \frac{c}{a} = \frac{0.3\times 10^9}{0.025} = 12 \text{ GHz}$$

Problem 2

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$TE_{01} \rightarrow \frac{c}{2b} = \frac{0.3\times 10^9}{2\times 0.01} = 15\text{ GHz}$$

$$TE_{11} \rightarrow \frac{c}{2}\sqrt{\left(\frac{1}{0.025}\right)^2 + \left(\frac{1}{0.01}\right)^2} = 16.155\text{ GHz}$$

$$TM_{11} \rightarrow \frac{c}{2}\sqrt{\left(\frac{1}{0.025}\right)^2 + \left(\frac{1}{0.01}\right)^2} = 16.155\text{ GHz}$$

TE_{30} (18 GHz), TE_{02} (30 GHz), TE_{21} (19.1 GHz) do not propagate

Problem 2

(b)

$$v_{pz} = \frac{\omega}{\beta_z} \Rightarrow \beta_z = \omega\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2 / f^2}$$

$$v_{pz} = \frac{c}{\sqrt{1 - f_c^2 / f^2}}$$

$$\text{At 10 GHz, } v_{pz} = \frac{0.3}{\sqrt{1-(6/10)^2}} = 0.375 \text{ m / ns}$$

$$\text{Time delay} = \frac{10}{0.375} = 26.66 \text{ ns}$$

Problem 2

$$\text{At 7 GHz, } v_{pz} = \frac{0.3}{\sqrt{1-(6/7)^2}} = 0.58243\ \text{m / ns}$$

$$\text{Time delay} = \frac{10}{0.582} = 17.169\ \text{ns}$$

Delay difference: 26.66 - 17.169 = 9.49 ns

The Lincoln Tunnel

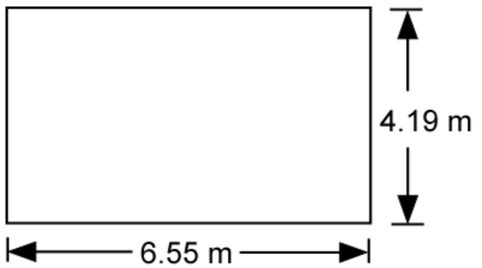
The Lincoln Tunnel is a 1.5 mile-long tunnel under the Hudson River. It connects Weehawken, New Jersey, to Midtown Manhattan in New York City on Route 495.

Width: 6.55 meters – Height: 4.19 meters

The Lincoln Tunnel

An AM radio station cannot be received inside the Lincoln tunnel. Why?

AM radio - 535 kilohertz to 1.7 megahertz



$$f_{cTE_{10}} = \frac{c}{2a} = \frac{0.3\times 10^9}{2\times 6.55} = 22.9\ \text{MHz}$$

AM signal will not propagate inside of tunnel!

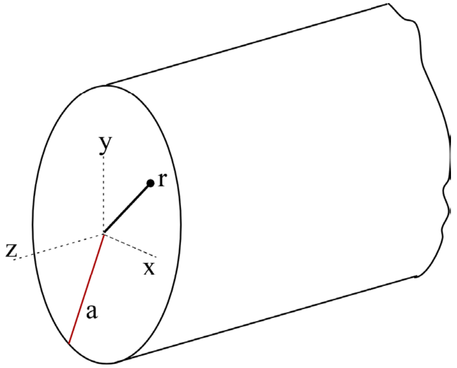
FM radio - 88 megahertz to 108 megahertz

FM radio will be received

Circular Waveguide - Fields

For a waveguide with arbitrary cross section, it is known that

TE Modes $$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}=\left[\beta_z^2-\omega^2\mu\varepsilon\right]H_z \quad \textbf{(1)}$$



TM Modes $$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}=\left[\beta_z^2-\omega^2\mu\varepsilon\right]E_z \quad \textbf{(2)}$$

We first assume TM modes in cylindrical coordinates:

$$\underbrace{\frac{\partial^2 E_z}{\partial r^2}+\frac{1}{r}\frac{\partial E_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2}}_{\nabla_{tr}^2 E_z}+\left(\gamma^2+\omega^2\mu\varepsilon\right)E_z=0$$

$$\gamma=\pm j\beta_z$$

See Reference [6].

Circular Waveguide – TM Modes

Solution will be in the form

$$E_z(r,\phi) = f(r)g(\phi)$$

Which after substitution gives

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + h^2 r^2 = -\frac{1}{g}\frac{d^2 g}{d\phi^2} \qquad \textbf{(3)}$$

where $h^2 = \gamma^2 + \omega^2 \mu\varepsilon$

For equality in (3) to hold, both sides must be equal to the same constant say n^2 where n is an integer in view of the azimuthal symmetry since the fields must be periodic in ϕ.

Circular Waveguide – TM Modes

$$\frac{d^2 g}{d\phi^2} + n^2 g = 0 \qquad \textbf{(4)}$$

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\left(h^2 - \frac{n^2}{r^2}\right) f = 0 \qquad \textbf{(5)}$$

Solution of (4) is of the form

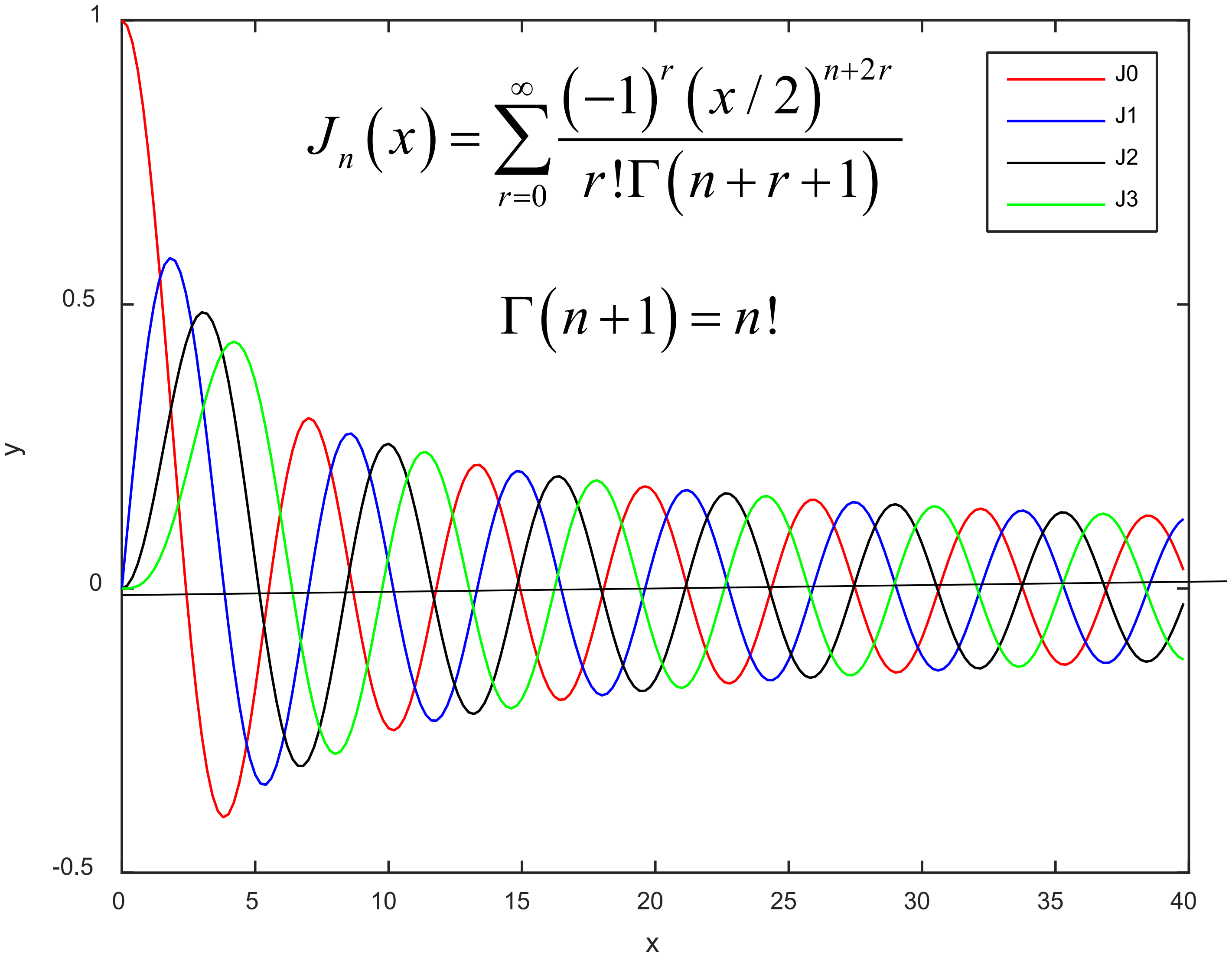
$$g(\phi) = C_1 \cos(n\phi) + C_2 \sin(n\phi) \qquad \textbf{(6)}$$

(5) is Bessel's equation and has solution

$$f(r) = C_3 J_n(hr) + C_4 Y_n(hr) \qquad \textbf{(7)}$$

J_n and Y_n are the n^{th} order Bessel functions of the first and second kinds respectively

Bessel Functions of the First Kind



$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$\Gamma(n+1) = n!$$

Circular Waveguide – TM Modes

Y_n has singularity at 0 and must consequently be discarded → $C_4 = 0$. The general solution then becomes

$$E_z(r,\phi) = C_3 J_n(hr)\left[C_1\cos(n\phi) + C_2\sin(n\phi)\right]$$

Since the origin for ϕ is arbitrary, the expression can be written as:

$$E_z(r,\phi) = C_n J_n(hr)\cos(n\phi)$$

where C_n is a constant. The boundary condition $E_{tan} = 0$ requires that

$$E_z(r,\phi) = 0 \ \ for\ r = a$$

Solution exists for only discrete values of h such that

$$J_n(ha) = 0$$

Circular Waveguide – TM Modes

ha **must be a root of the** n^{th} **order Bessel function. If we assume that** t_{nl} **is the** l^{th} **root of** J_n**, we can define a set of eigenvalues** h_{nl} **for the TM modes so that:**

$$h_{TM_{nl}} = \frac{t_{nl}}{a}$$

l^{th} **root of** $J_n(.)=0$

n →, l ↓

	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	13.323	11.620

Each choice of n **and** l **specifies a particular solution or** ***mode***

n **is related to the number of circumferential variations and** l **describes the number of radial variations of the field.**

Circular Waveguide – TM Modes

The propagation constant of the nl^{th} propagating TM mode is:

$$\beta_{TM_{nl}} = \left[\omega^2\mu\varepsilon - \left(\frac{t_{nl}}{a}\right)^2\right]^{1/2}$$

The propagation occurs for $\lambda < \lambda_{cTMnl}$ or $f > f_{cTMnl}$ where the cutoff frequency and wavelength can be found from $\gamma = 0$ as:

$$\lambda_{cTMnl} = \frac{2\pi a}{t_{nl}} \qquad\qquad f_{cTMnl} = \frac{t_{nl}}{2\pi a\sqrt{\mu\varepsilon}}$$

The other field components can be obtained from E_z

$$E_z = C_n J_n\left(\frac{t_{nl}}{a}r\right)\cos(n\phi)e^{-j\beta_{nl}z}$$

Circular Waveguide – TE Modes

The solutions for the TE modes can be found in a similar manner except that we solve for $H_z(r,\phi)$ to get:

$$H_z(r,\phi) = C_n J_n(hr)\cos(n\phi)$$

To apply the boundary condition $E_{tan} = 0$, we require

$$\frac{\partial H_z}{\partial r} \text{ to be 0 at } r = a$$

We must have $\hat{n} \cdot \nabla_{tr} H_z = \dfrac{\partial H_z}{\partial r} = 0 \quad at \quad r = a$

For this, we need the zeros of $J_n'(u)$ given by s_{nl}. The propagation constant, cutoff frequency and wavelength have the same expressions as in the TM case with $t_{nl} \rightarrow s_{nl}$.

Circular Waveguide – TE Modes

The propagation constant of the *nl^th^* propagating TE mode is:

$$\beta_{TE_{nl}} = \left[\omega^2 \mu\varepsilon - \left(\frac{s_{nl}}{a} \right)^2 \right]^{1/2}$$

l^{th} root of $J_n'(.)=0$

$n \rightarrow$, $l \downarrow$

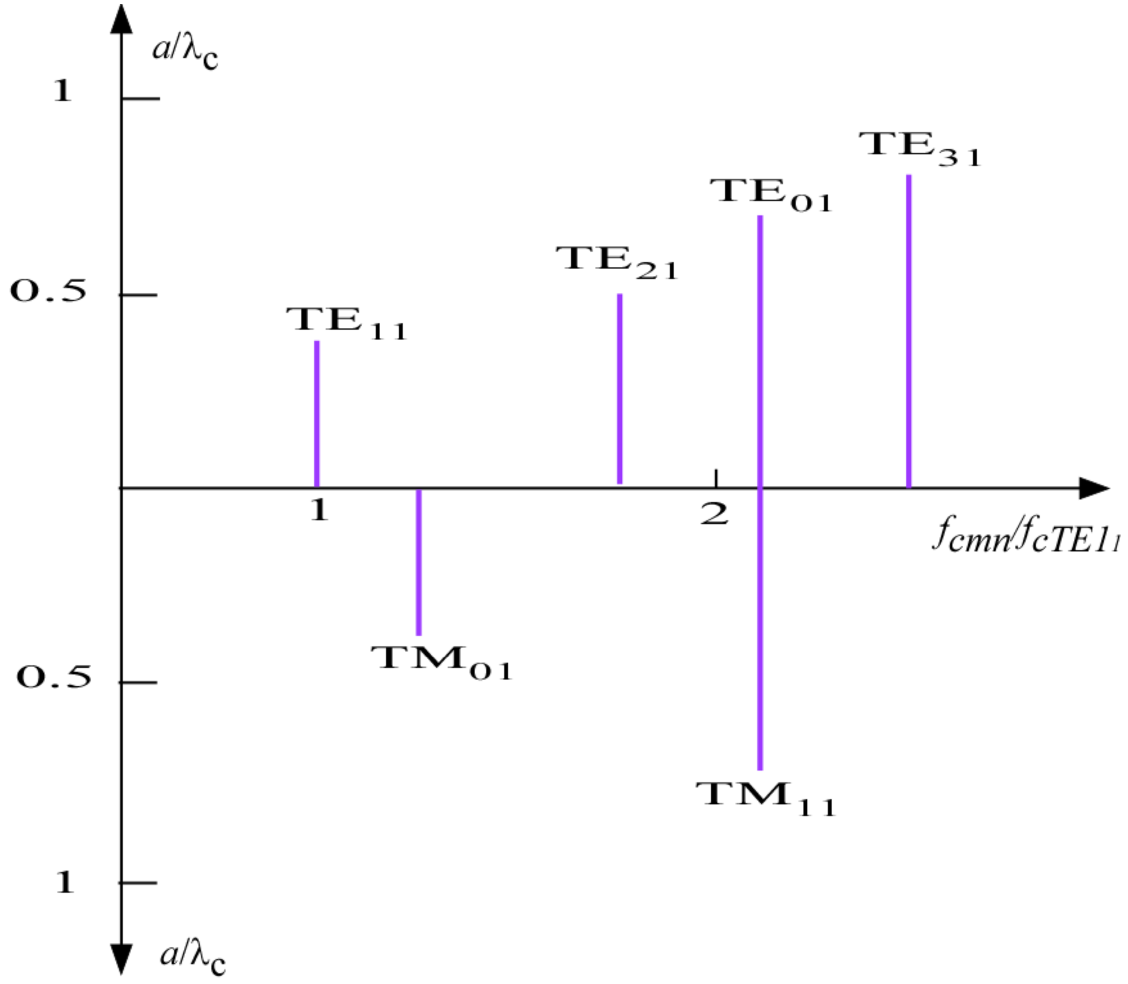
	0	1	2
1	3.832	1.841	3.054
2	7.016	5.331	6.706
3	10.173	8.536	9.969

From the tables, it can be seen that the lowest cutoff frequency is the TE_{11} mode.

and for TE modes,

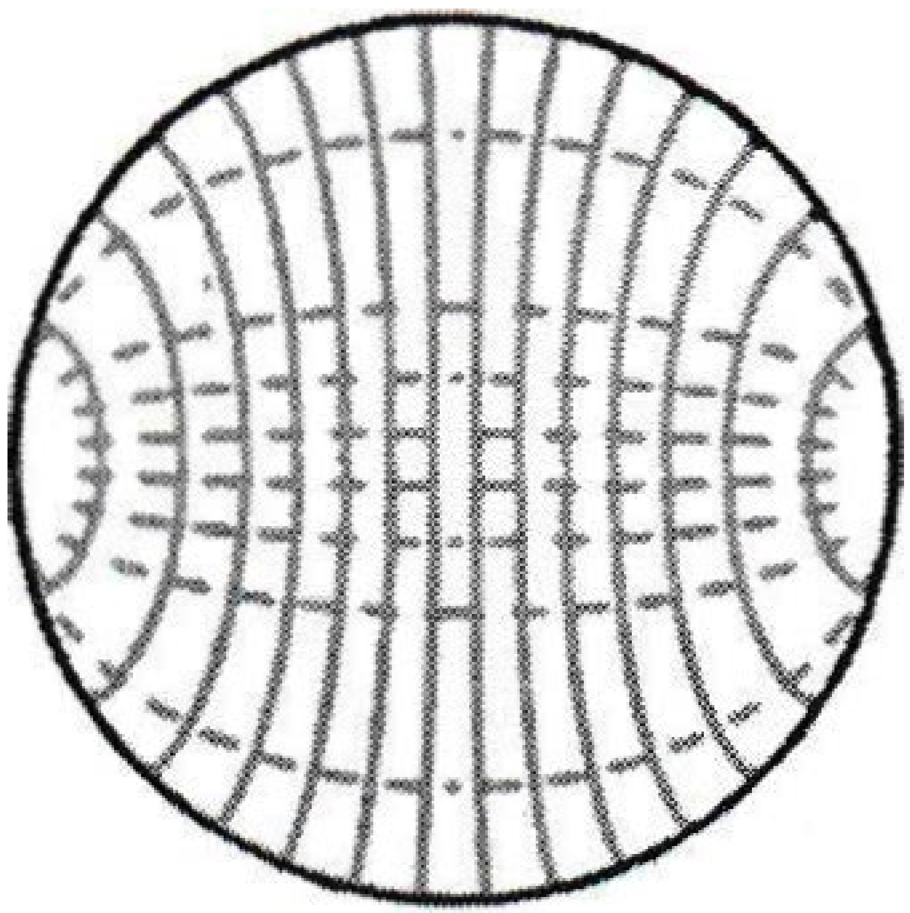
$$H_z = C_n J_n\left(\frac{s_{nl}}{a} r \right) \cos(n\phi) e^{-j\beta_{nl} z}$$

Circular Waveguide – TE & TM Modes



See Reference [6].

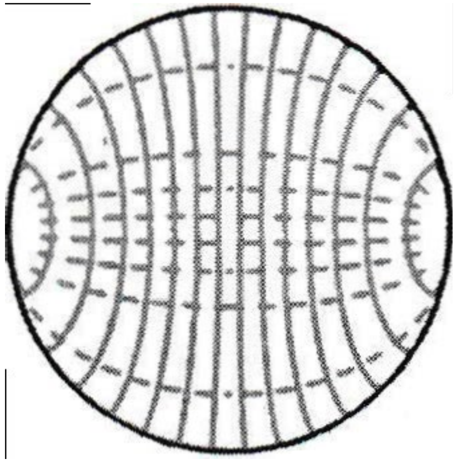
TE_{11} Mode in Circular Waveguide



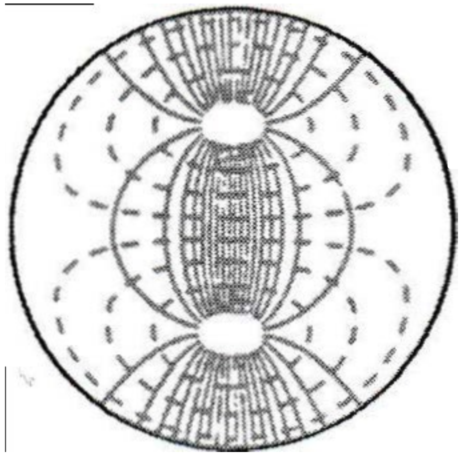
E
H

See Reference [1].

Modes in Circular Waveguide



TE_{11}

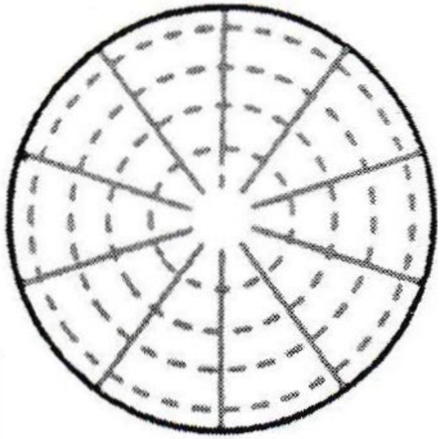


TM_{11}

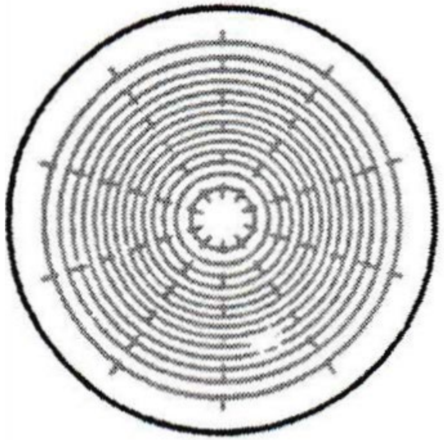
E ———

H - - - - -

See Reference [1].



TM_{01}



TE_{01}

Example: Circular Waveguide Design

Design an air-filled circular waveguide such that only the dominant mode will propagate over a bandwidth of 10 GHz.

Solution: the cutoff frequency of the TE_{11} mode is the lower bound of the bandwidth.

$$f_{cTE_{11}} = \frac{1.8412c}{2\pi a}$$

The next mode is the TM_{01} with cutoff frequency:

$$f_{cTM_{01}} = \frac{2.4049c}{2\pi a}$$

Example: Circular Waveguide Design

The BW is the difference between these two frequencies

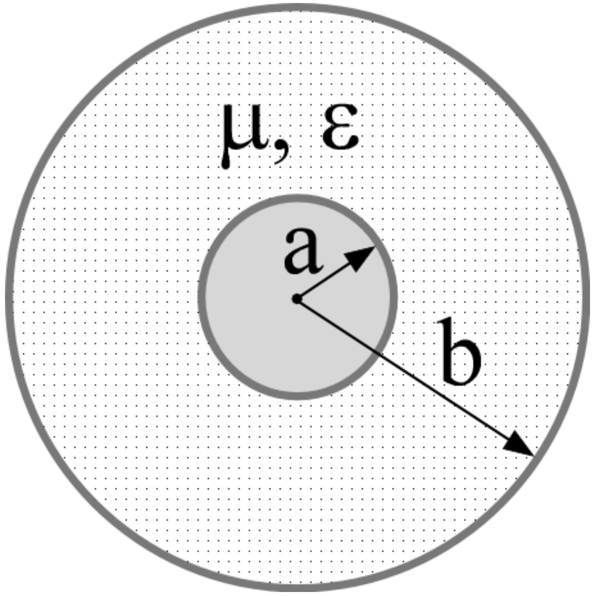
$$BW = f_{cTM_{01}} - f_{cTE_{11}} = \frac{c}{2\pi a}\left(2.4049 - 1.8412\right) = 10 GHz$$

From which we find *a* = 0.269 cm

So that

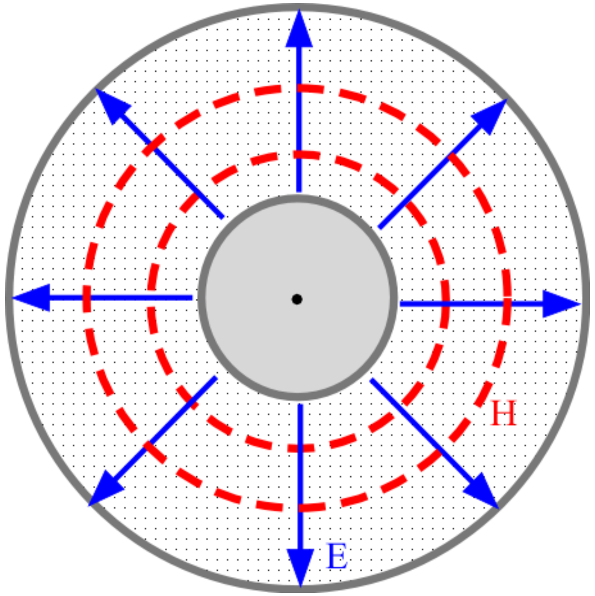
$$f_{cTE_{11}} = 32.7\ GHz\ and\ f_{cTM_{11}} = 42.76\ GHz$$

Coaxial Waveguide



- Most common two-conductor transmission system
- Dielectric filling in most microwave applications is polyethylene or Teflon

Coaxial Waveguide – TEM Mode



- Two-conductor system → Dominant mode is TEM
- Tangential E-field and normal H field must be 0 in conductor surfaces

$$E_\phi = 0 \text{ and } H_r = 0 \text{ at } r = a, b$$

Coaxial Waveguide – TEM Mode

TEM solution can exist only with

$$E = \hat{r}E_r\left(r,z\right) \quad \text{and} \quad H = \hat{\phi}H_\phi\left(r,z\right)$$

with no ϕ dependence because of azimuthal symmetry

we get

$$-\frac{\partial H_\phi}{\partial z} = j\omega E_r \rightarrow j\beta H_\phi^o\left(r\right) = j\omega\varepsilon E_r^o\left(r\right)$$

$$-\frac{1}{r}H_\phi + \frac{\partial H_\phi}{\partial r} = 0 \rightarrow -\frac{1}{r}H_\phi^o\left(r\right) + \frac{\partial H_\phi^o}{\partial r} = 0$$

Where propagation in z direction is assumed.

Coaxial Waveguide – TEM Mode

We get

$$\mathbf{H} = \hat{\phi}\frac{H_o}{r}e^{-j\beta z} \qquad \mathbf{E} = \hat{r}\frac{H_o\eta}{r}e^{-j\beta z}$$

where H_o is a constant. No cutoff condition for TEM mode.

The voltage between the two conductors is given by

$$V(z) = -\eta H_o \ln(b/a)e^{-j\beta z}$$

The current in the inner conductor is given by

$$I(z) = 2\pi H_o e^{-j\beta z}$$

The characteristic impedance Z_o is thus given by

$$Z_o = \eta\frac{\ln(b/a)}{2\pi}$$

Coaxial Waveguide – TE and TM Modes

TE and TM modes may also exist in addition to TEM. In a coaxial line, they are generally undesirable.

For TM modes, we have:

$$E_z^o\left(r,\phi\right)=\left[C_3J_n\left(hr\right)+C_4Y_n\left(hr\right)\right]\cos\left(n\phi\right)$$

For TE modes, we have:

$$H_z^o\left(r,\phi\right)=\left[C_3^{'}J_n\left(hr\right)+C_4^{'}Y_n\left(hr\right)\right]\cos\left(n\phi\right)$$

With boundary conditions at *r* =*a, b* of

$$E_z\left(r,\phi\right)=0 \quad \text{for TM modes}$$

$$\frac{\partial H_z}{\partial r}=0 \quad \text{for TE modes}$$

Coaxial Waveguide – TE and TM Modes

These conditions lead to

$$J_n(ha)Y_n(hb) = J_n(hb)Y_n(ha) \quad \text{for TM modes}$$

$$J_n'(ha)Y_n'(hb) = J_n'(hb)Y_n'(ha) \quad \text{for TE modes}$$

Solutions of these transcendental equations determine the eigenvalues of *h* for given *a, b*. As in the circular waveguide case, the modes for coaxial waveguide are denoted TE_{nl} and TM_{nl}.

Coaxial Waveguide – TE and TM Modes

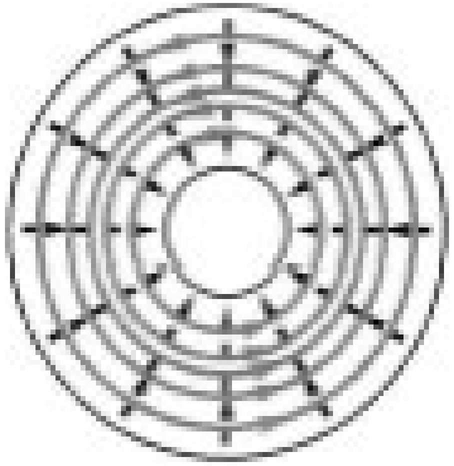
The mode with the lowest cutoff frequency is the TE_{11} mode for which the eigenvalue *h* is approximated as:

$$h = \frac{2}{a+b}$$

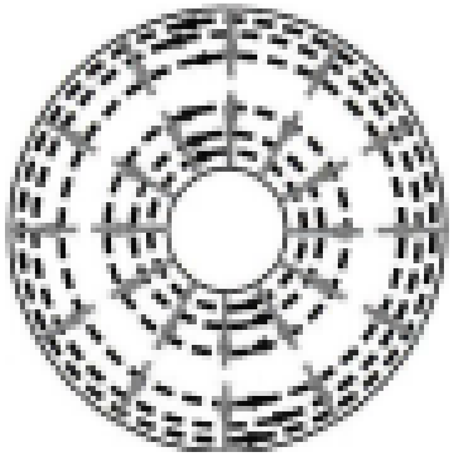
The cutoff frequency and cutoff wavelength are given by

$$\lambda_{c11} = \frac{2\pi}{h} \simeq \pi\left(a+b\right) \quad \text{and} \quad f_{c11} \simeq \frac{1}{\pi\left(a+b\right)\sqrt{\mu\varepsilon}}$$

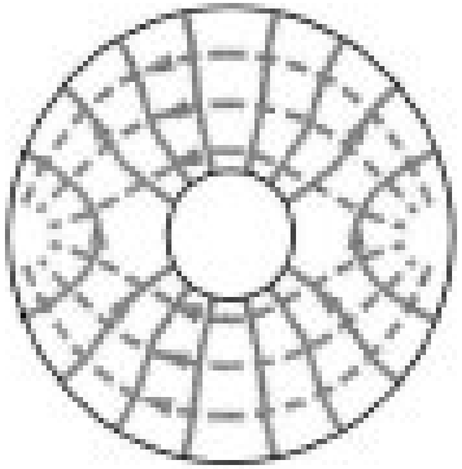
Coaxial Waveguide – TE and TM Modes



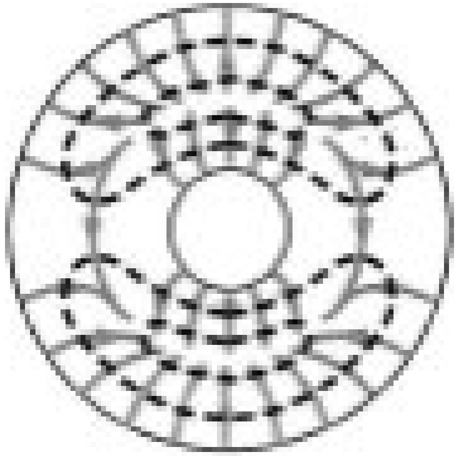
TE_{01}



TM_{01}



TE_{11}



TM_{11}

See Reference [3].

References

[1]**. C. S. Lee, S. W. Lee, and S. L. Chuang**, "Plot of modal field distribution in rectangular and circular waveguides", *IEEE Trans. Microwave Theory and Techniques*, 33(3), pp. 271-274, March 1985.

[2]. **J. H. Bryant,** "Coaxial transmission lines, related two-conductor transmission lines, connectors, and components: A U.S. historical perspective", *IEEE Trans. Microwave Theory and Techniques,* 32(9), pp. 970-983, September 1984.

[3]. **H. A. Atwater**, "*Introduction to Microwave Theory*", p. 76, McGraw-Hill, New York, 1962.

[4]. **N. Marcuvitz**, "*Waveguide Handbook*", IEEE Press, Piscataway, New Jersey, 1986.

[5]. **S. Ramo, J. R. Whinnery, and T. Van Duzer**, "*Fields and Waves in Communication Electronics*", John Wiley & Sons, New York, 1994.

[6]. **U. S. Inan and A. S. Inan**, "*Electromagnetice Waves*", Prentice Hall, 2000.