

ECE 546

Lecture - 05

Transmission Lines

Spring 2024

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Maxwell's Equations

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

Faraday's Law of Induction

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

Ampère's Law

$$\nabla \cdot D = \rho$$

Gauss' Law for electric field

$$\nabla \cdot B = 0$$

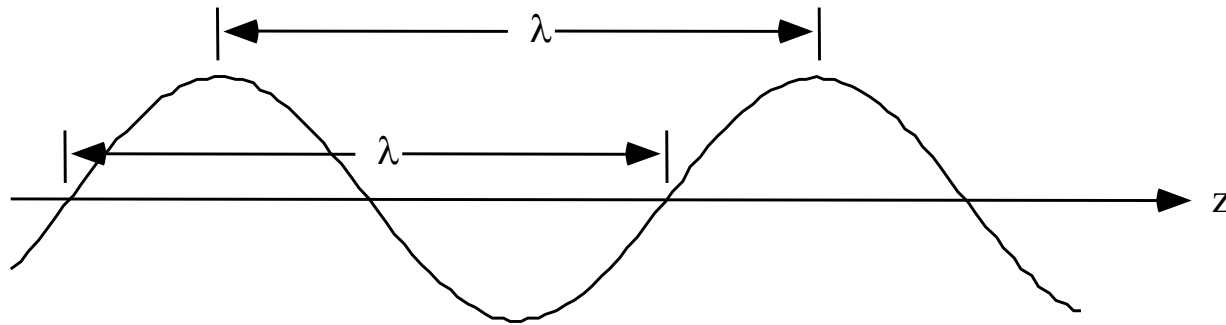
Gauss' Law for magnetic field

Constitutive Relations

$$B = \mu H$$

$$D = \epsilon E$$

Why Transmission Line?



Wavelength : λ

$$\lambda = \frac{\text{propagation velocity}}{\text{frequency}}$$

Why Transmission Line?

In Free Space

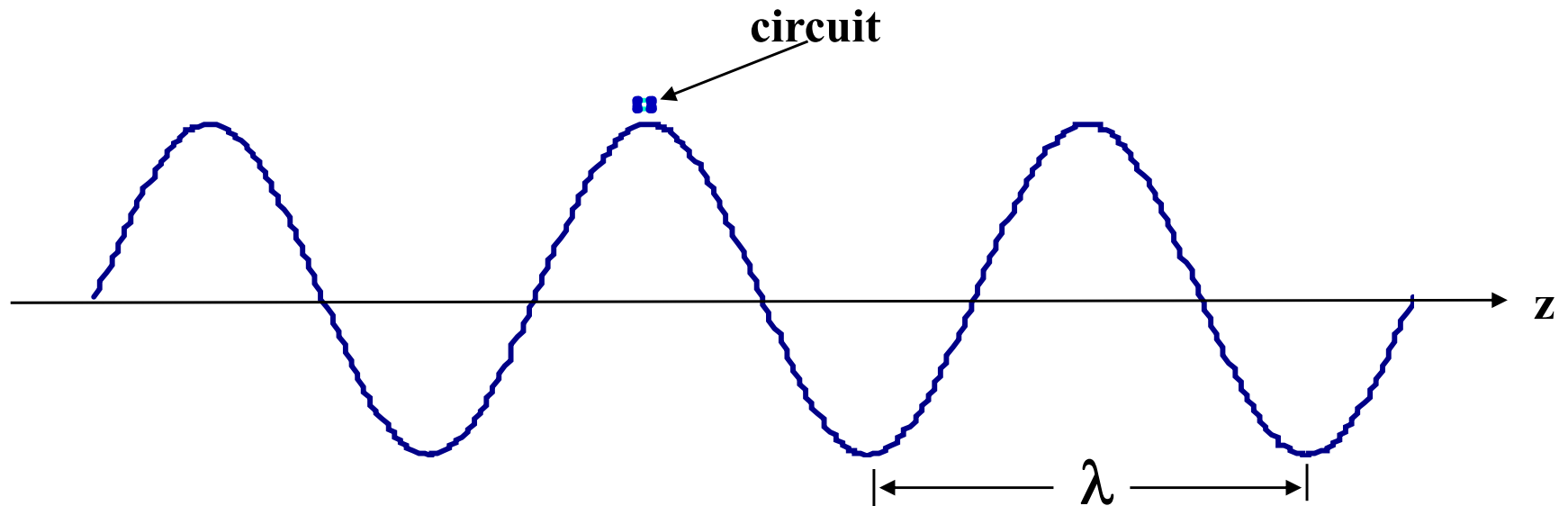
At 10 KHz : $\lambda = 30$ km

At 10 GHz : $\lambda = 3$ cm

Transmission line behavior is prevalent when the structural dimensions of the circuits are comparable to the wavelength.

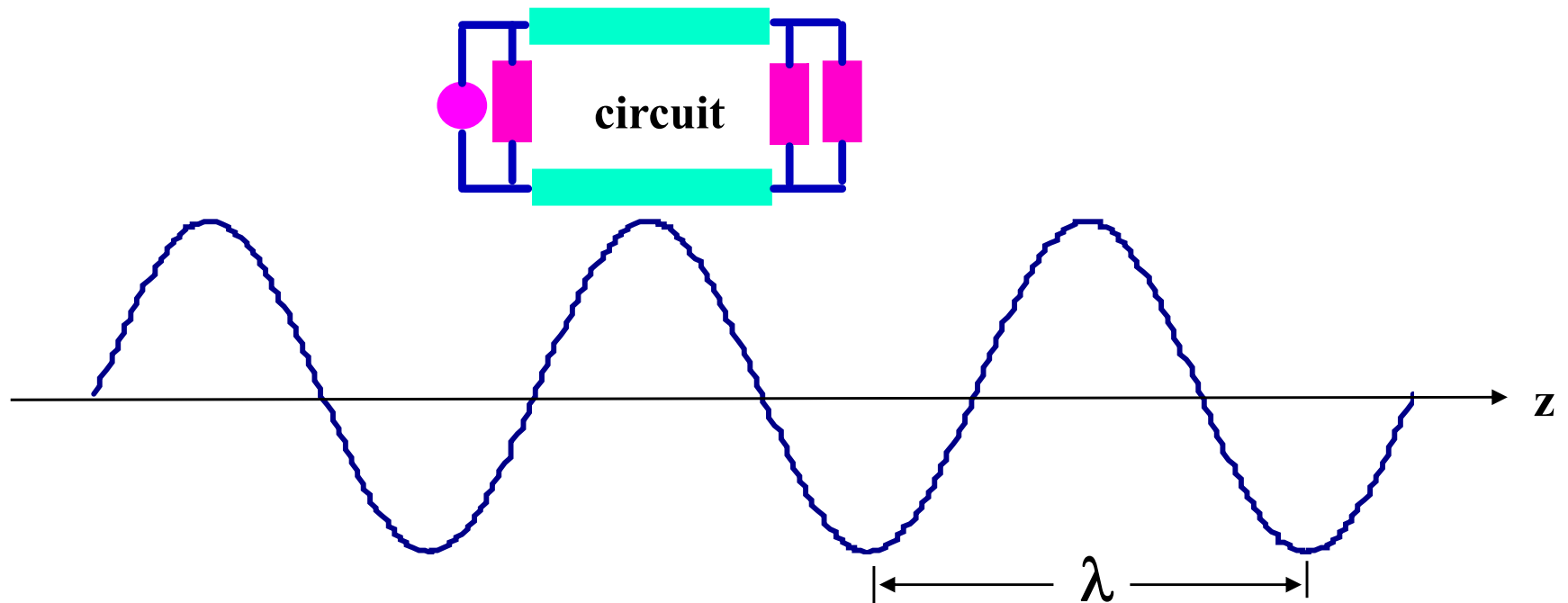
Justification for Transmission Line

Let d be the largest dimension of a circuit



If $d \ll \lambda$, a lumped model for the circuit can be used

Justification for Transmission Line



If $d \approx \lambda$, or $d > \lambda$ then use transmission line model

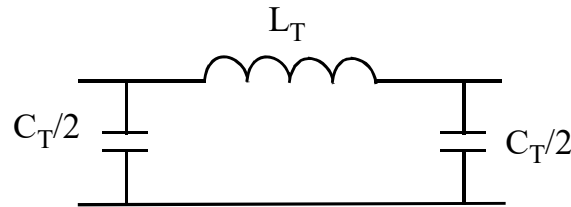
Modeling Interconnections

Low Frequency

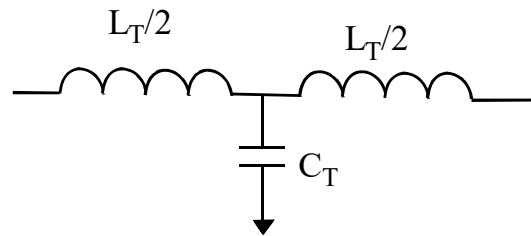


Short

Mid-range
Frequency

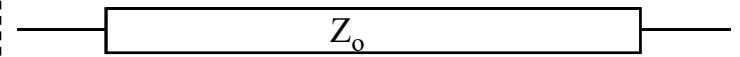


or



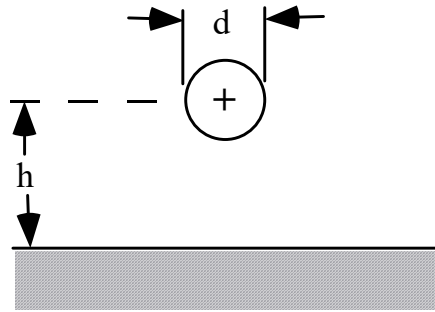
Lumped
Reactive CKT

High Frequency



Transmission
Line

Single wire near ground



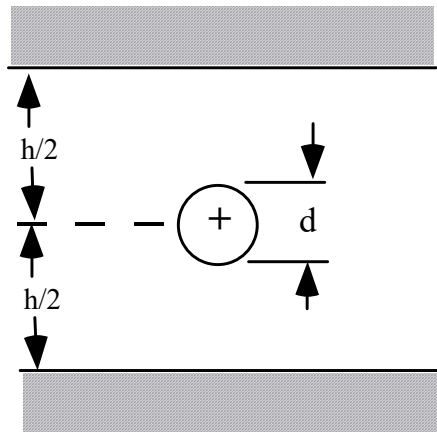
$$\text{for } d \ll h, \quad Z_o = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{4h}{d}\right)$$

$$Z_o = 120 \cosh^{-1}(D/d)$$

$$C = \frac{2\pi\epsilon}{\ln\left(\frac{4h}{d}\right)}$$

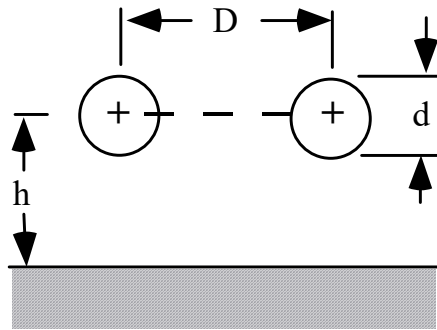
$$L = \frac{\mu}{2\pi} \ln \frac{4h}{d}$$

Single wire between grounded parallel planes ground return



$$Z_o = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln \left(\frac{4h}{\pi d} \right)$$

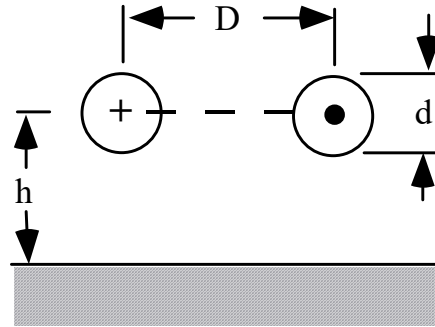
Wires in parallel near ground



for $d \ll D, h$

$$Z_o = \left(69 / \sqrt{\epsilon_r}\right) \log_{10} \left\{ (4h/d) \sqrt{1 + (2h/D)^2} \right\}$$

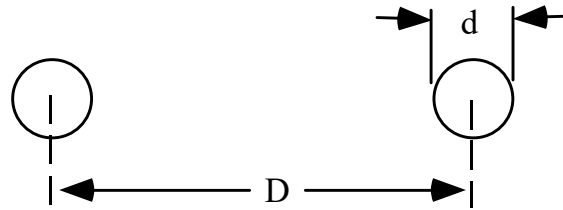
Balanced, near ground



for $d \ll D, h$

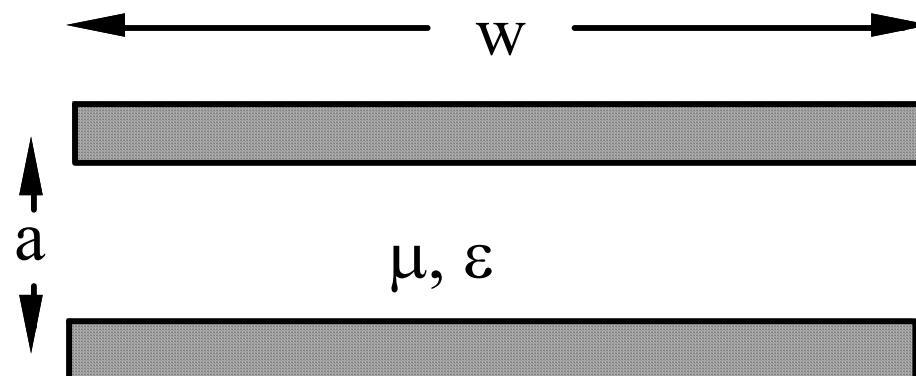
$$Z_0 = \left(276 / \sqrt{\epsilon_r}\right) \log_{10} \left\{ \frac{(2D/d)}{\sqrt{1 + (D/2h)^2}} \right\}$$

Open 2-wire line in air



$$Z_o = 120 \cosh^{-1}(D/d)$$

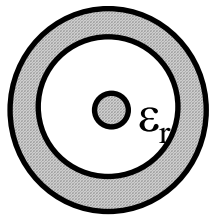
Parallel-plate Transmission Line



$$\mathbf{L} = \frac{\mu \mathbf{a}}{\mathbf{w}}$$

$$\mathbf{C} = \frac{\epsilon \mathbf{w}}{\mathbf{a}}$$

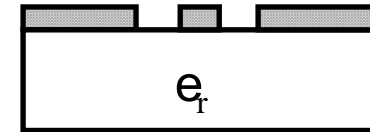
Types of Transmission Lines



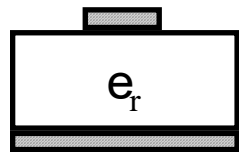
Coaxial line



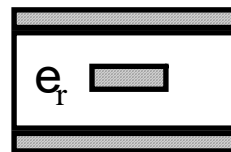
Waveguide



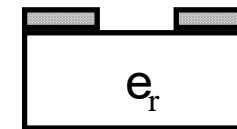
Coplanar line



Microstrip

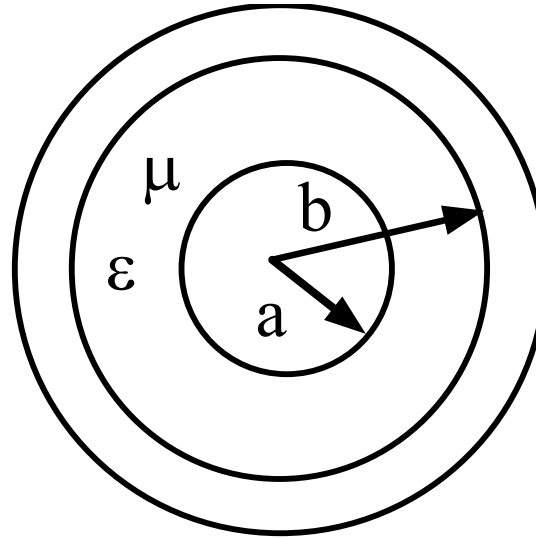


Stripline



Slot line

Coaxial Transmission Line

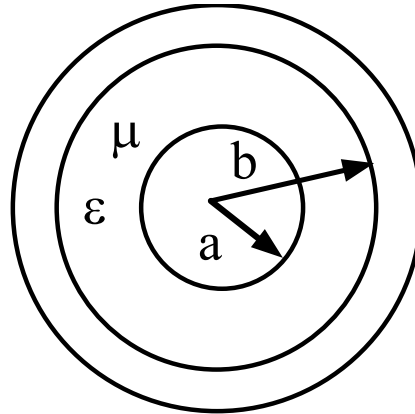


TEM Mode of Propagation

$$L = \mu \ln \frac{b}{a}$$

$$C = \frac{2\pi\epsilon}{\ln(b/a)}$$

Coaxial Air Lines



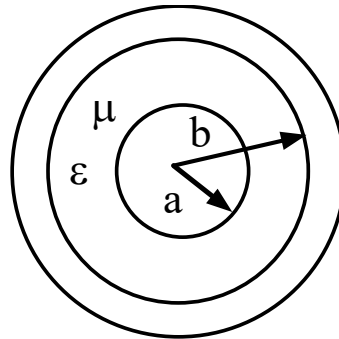
Infinite Conductivity

$$Z_o = \frac{\sqrt{\mu / \epsilon}}{2\pi} \ln(b / a)$$

Finite Conductivity

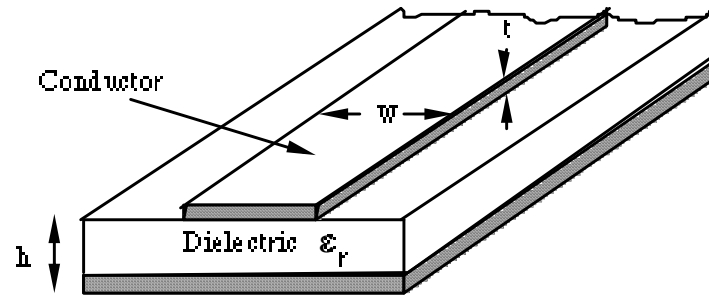
$$Z_o = \frac{\sqrt{\mu / \epsilon}}{2\pi} \ln(b / a) \left[1 + \frac{(1/a + 1/b)}{4\sqrt{\pi f \mu \sigma} \ln(b / a)} (1 - j) \right]$$

Coaxial Connector Standards

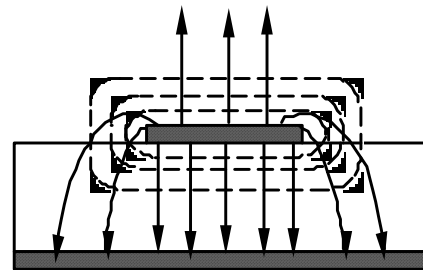


<u>Connector</u>	<u>Frequency Range</u>
14 mm	DC - 8.5 GHz
GPC-7	DC - 18 GHz
Type NDC	- 18 GHz
3.5 mm	DC - 33 GHz
2.92 mm	DC - 40 GHz
2.4 mm	DC - 50 GHz
1.85 mm	DC - 65 GHz
1.0 mm	DC - 110 GHz

Microstrip



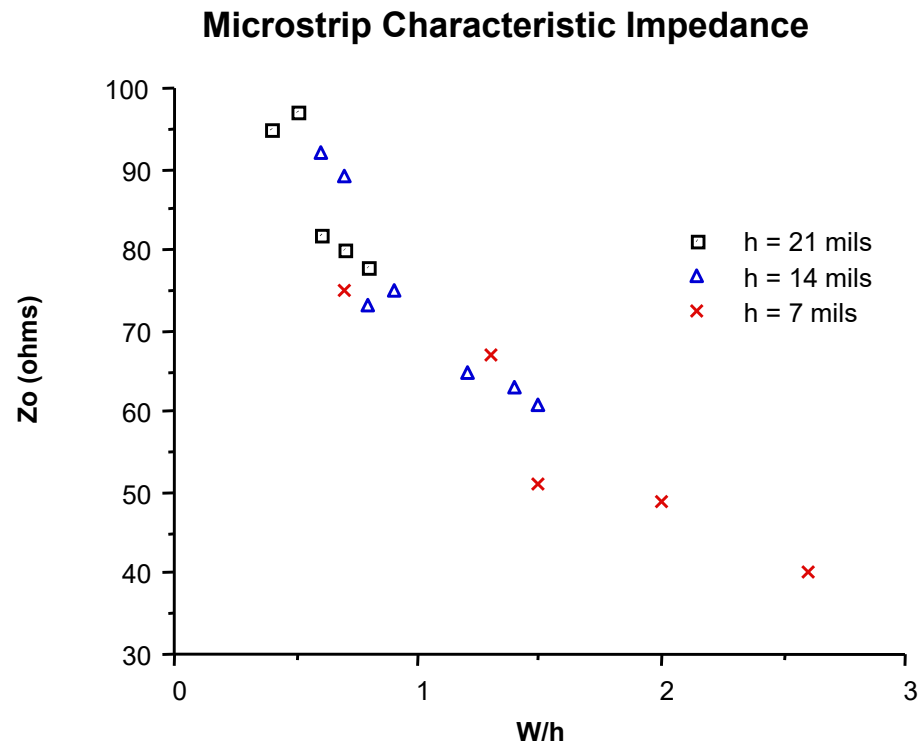
(a)



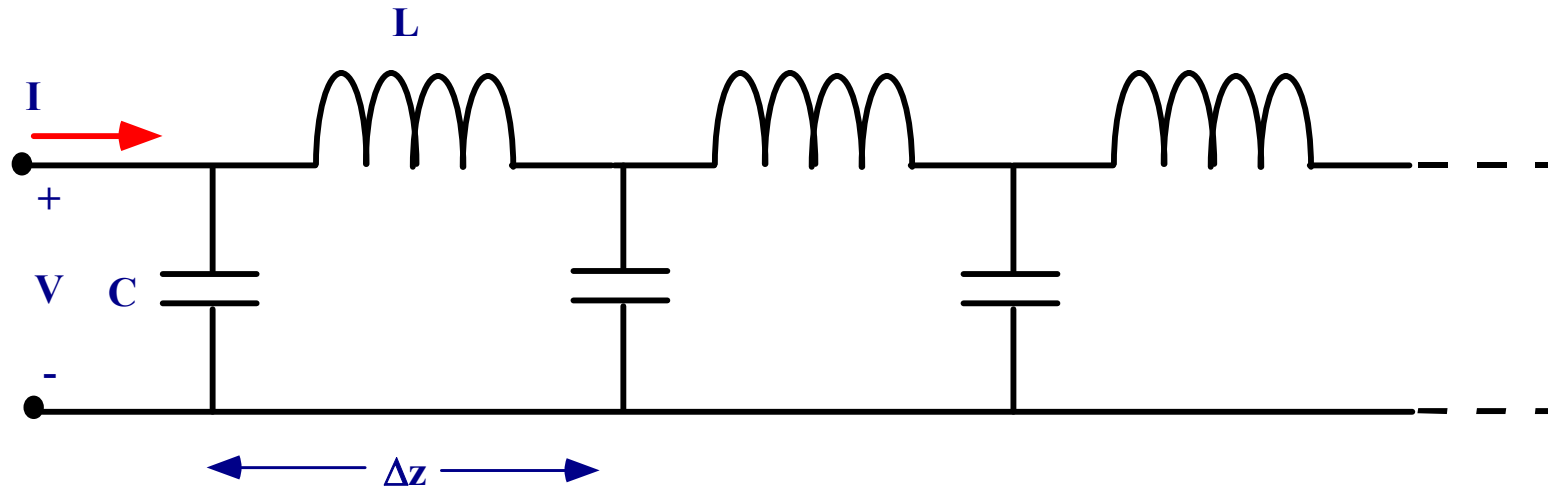
————— Electric field lines
- - - - - Magnetic field lines

(b)

Microstrip



Telegraphers' Equations



L: Inductance per unit length.

C: Capacitance per unit length.

$$-\frac{\partial V}{\partial z} = L \frac{\partial I}{\partial t}$$

Assume
time-harmonic
dependence

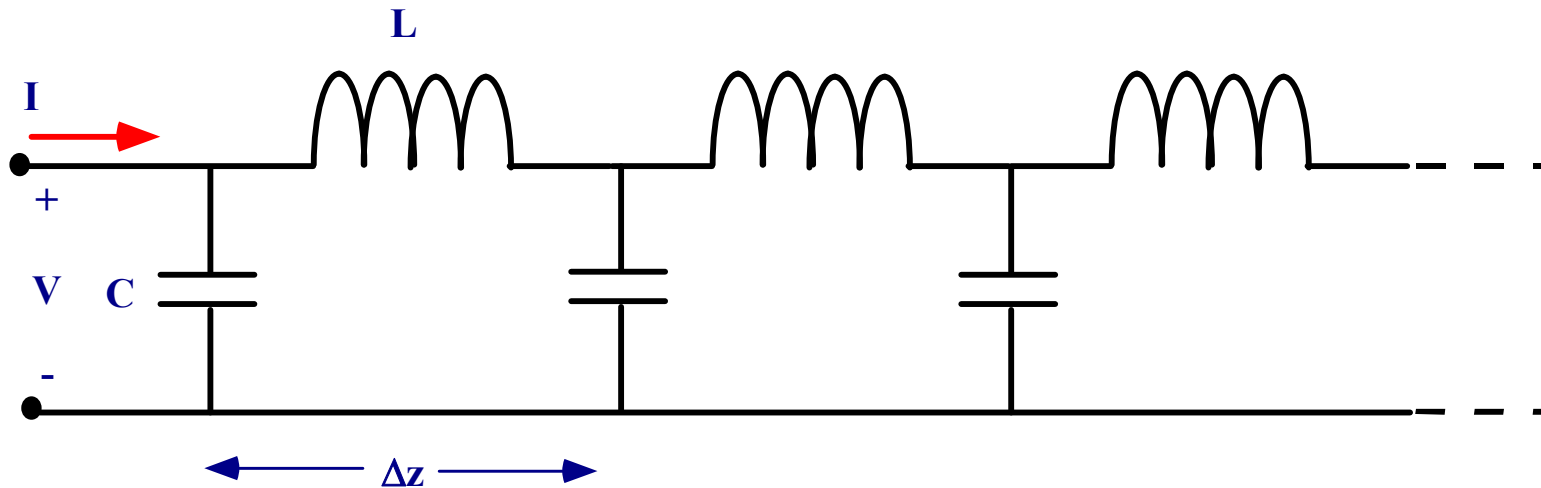
$$-\frac{\partial V}{\partial z} = j\omega LI$$

$$\frac{\partial I}{\partial z} = C \frac{\partial V}{\partial t}$$

$$V, I \sim e^{j\omega t}$$

$$\frac{\partial I}{\partial z} = j\omega CV$$

TL Solutions

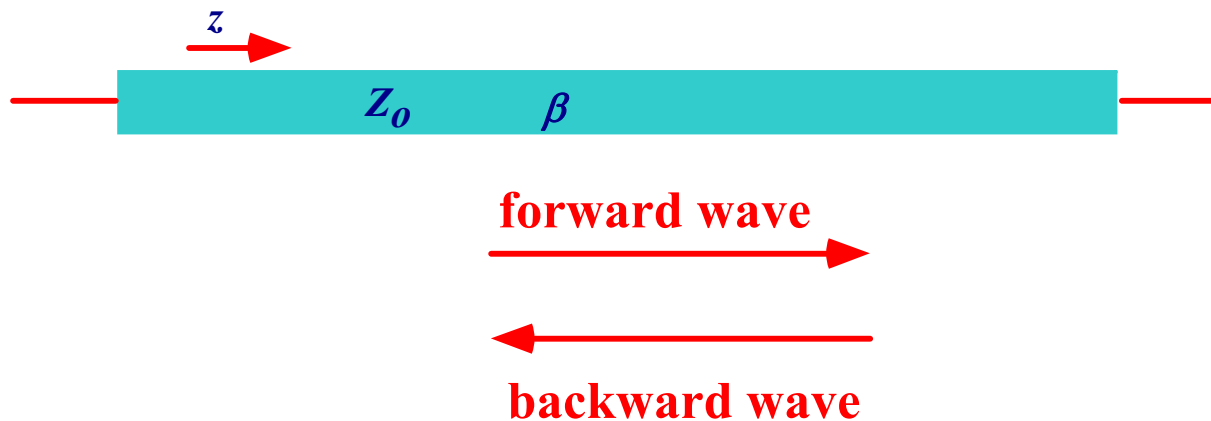


$$-\frac{\partial}{\partial z} \left(\frac{\partial V}{\partial z} \right) = -\frac{\partial^2 V}{\partial z^2} = j\omega L \frac{\partial I}{\partial z} = -j\omega L j\omega C V \quad \Rightarrow \quad \frac{\partial^2 V}{\partial z^2} = -\omega^2 L C V$$

$$-\frac{\partial}{\partial z} \left(\frac{\partial I}{\partial z} \right) = -\frac{\partial^2 I}{\partial z^2} = j\omega C \frac{\partial V}{\partial z} = -j\omega L j\omega C I \quad \Rightarrow \quad \frac{\partial^2 I}{\partial z^2} = -\omega^2 C L I$$

TL Solutions

(Frequency Domain)



$$\beta = \omega\sqrt{LC}$$

$$V(z) = \overbrace{V_+ e^{-j\beta z}}^{\text{Forward Wave}} + \overbrace{V_- e^{+j\beta z}}^{\text{Backward Wave}}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$I(z) = \underbrace{\frac{V_+}{Z_0} e^{-j\beta z}}_{\text{Forward Wave}} - \underbrace{\frac{V_-}{Z_0} e^{+j\beta z}}_{\text{Backward Wave}}$$

TL Solutions

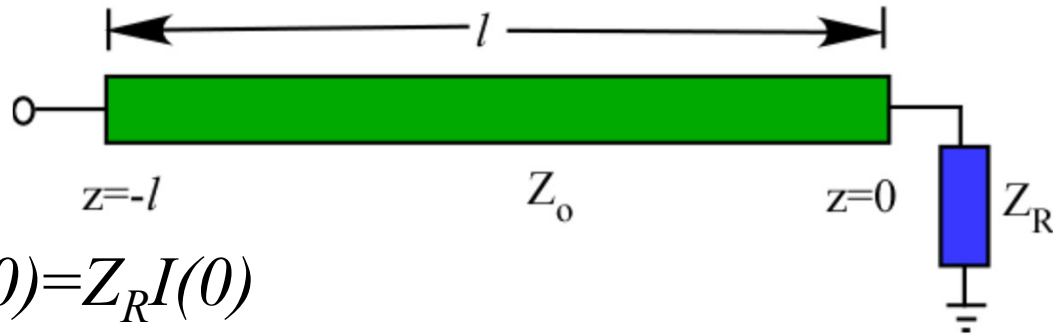
Propagation constant $\beta = \omega\sqrt{LC}$ **Propagation velocity** $v = \frac{1}{\sqrt{LC}}$

Characteristic impedance $Z_o = \sqrt{\frac{L}{C}}$ **Wavelength** $\lambda = \frac{v}{f}$

$$V(z, t) = \overbrace{V_+ \cos(\omega t - \beta z)}^{\text{Forward Wave}} + \overbrace{V_- \cos(\omega t + \beta z)}^{\text{Backward Wave}}$$

$$I(z, t) = \underbrace{\frac{V_+}{Z_o} \cos(\omega t - \beta z)}_{\text{Forward Wave}} - \underbrace{\frac{V_-}{Z_o} \cos(\omega t + \beta z)}_{\text{Backward Wave}}$$

Reflection Coefficient



At $z=0$, we have $V(0)=Z_R I(0)$

But from the TL equations:

$$\begin{aligned} V(0) &= V_+ + V_- \\ I(0) &= \frac{V_+}{Z_o} - \frac{V_-}{Z_o} \end{aligned} \quad \frac{Z_R}{Z_o} (V_+ - V_-) = V_+ + V_-$$

Which gives $V_- = \Gamma_R V_+$

where $\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$ is the load reflection coefficient

Reflection Coefficient

- If $Z_R = Z_o$, $\Gamma_R = 0$, no reflection, the line is matched
- If $Z_R = 0$, short circuit at the load, $\Gamma_R = -1$
- If $Z_R \rightarrow \text{inf}$, open circuit at the load, $\Gamma_R = +1$

V and I can be written in terms of Γ_R

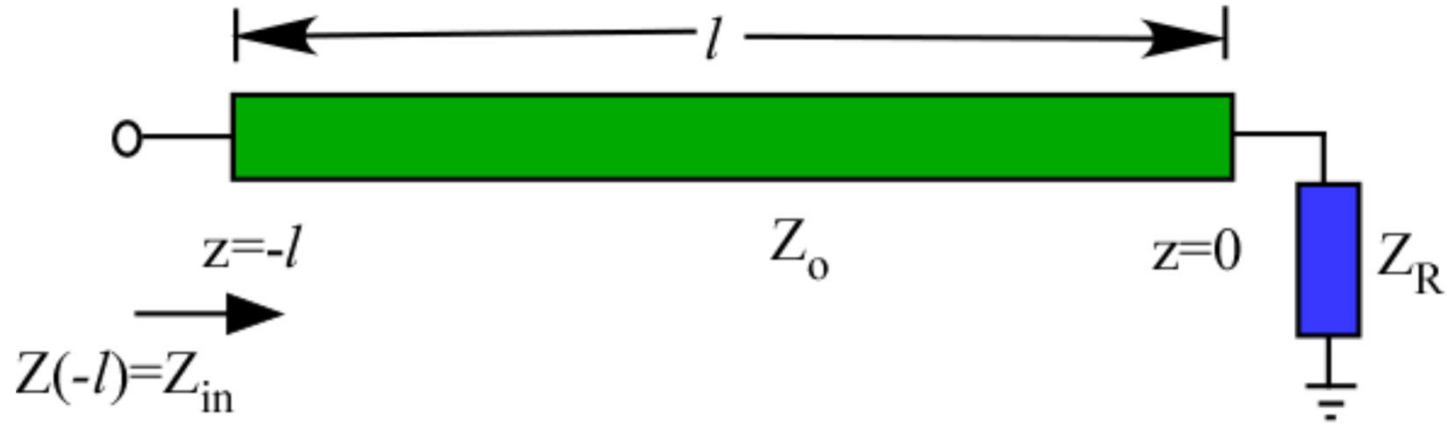
$$V(z) = V_+ \left[e^{-j\beta z} + \Gamma_R e^{+j\beta z} \right]$$

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

$$I(z) = \frac{V_+}{Z_o} \left[e^{-j\beta z} - \Gamma_R e^{+j\beta z} \right]$$

$$I(z) = \frac{V_+ e^{-j\beta z}}{Z_o} \left[1 - \Gamma_R e^{+2j\beta z} \right]$$

Generalized Impedance

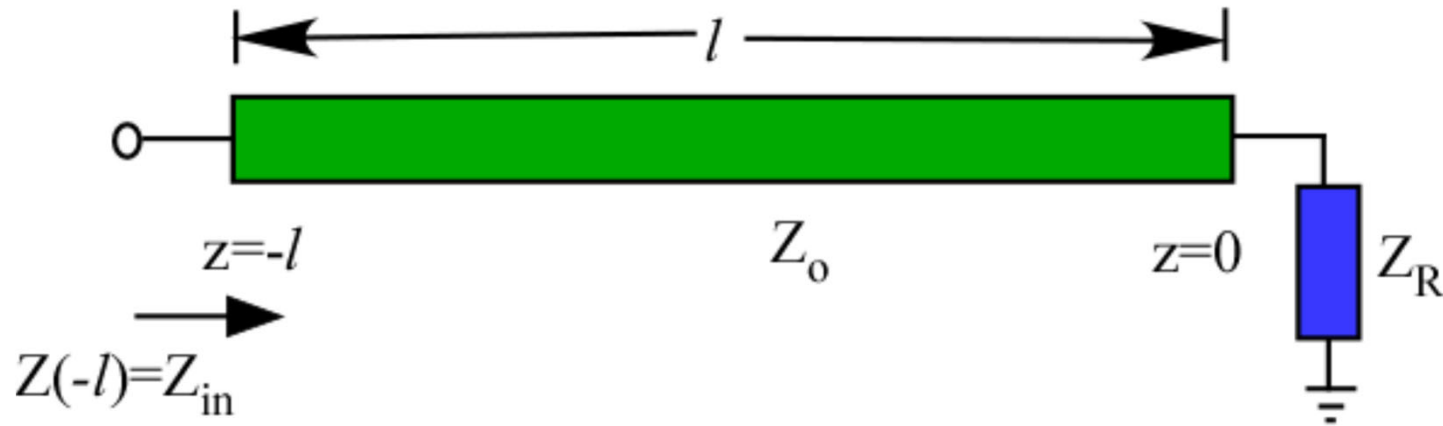


$$Z(z) = \frac{V(z)}{I(z)} = Z_o \left[\frac{e^{-j\beta z} + \Gamma_R e^{+j\beta z}}{e^{-j\beta z} - \Gamma_R e^{+j\beta z}} \right]$$

$$Z(-l) = Z_o \left[\frac{Z_R + jZ_o \tan \beta l}{Z_o + jZ_R \tan \beta l} \right]$$

Impedance
transformation
equation

Generalized Impedance



- Short circuit $Z_R = 0$, line appears inductive for $0 < l < \lambda/2$

$$Z(-l) = jZ_o \tan \beta l$$

- Open circuit $Z_R \rightarrow \text{inf}$, line appears capacitive for $0 < l < \lambda/2$

$$Z(-l) = \frac{Z_o}{j \tan \beta l}$$

- If $l = \lambda/4$, the line is a quarter-wave transformer

$$Z(-l) = \frac{Z_o^2}{Z_R}$$

Generalized Reflection Coefficient

$$\Gamma(z) = \frac{\text{Backward traveling wave at } z}{\text{Forward traveling wave at } z} = \frac{V_b(z)}{V_f(z)}$$

$$\Gamma(z) = \frac{V_- e^{+j\beta z}}{V_+ e^{-j\beta z}} = \frac{V_-}{V_+} e^{+2j\beta z} = \Gamma_R e^{+2j\beta z}$$

**Reflection coefficient
transformation equation**

➔ $\Gamma(-l) = \Gamma_R e^{-2j\beta l}$

$$Z(z) = Z_o \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

$$\Gamma(z) = \frac{Z(z) - Z_o}{Z(z) + Z_o}$$

Voltage Standing Wave Ratio (VSWR)

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

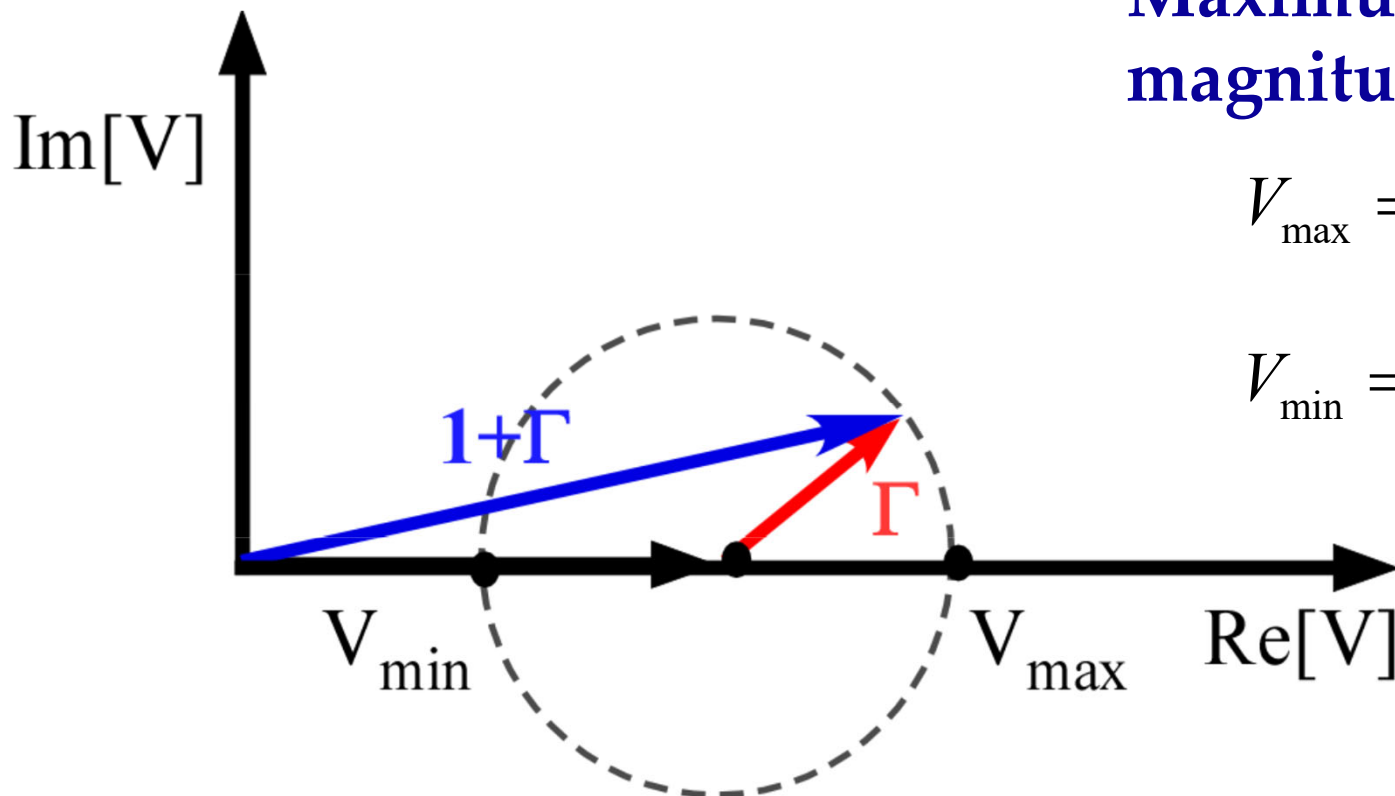
We follow the magnitude of the voltage along the TL

$$|V(z)| = |V_+ e^{-j\beta z}| \left| 1 + \Gamma_R e^{+2j\beta z} \right| = |V_+| \left| 1 + \Gamma_R e^{+2j\beta z} \right|$$

Maximum and minimum magnitudes given by

$$V_{\max} = \left[1 + |\Gamma_R| \right]$$

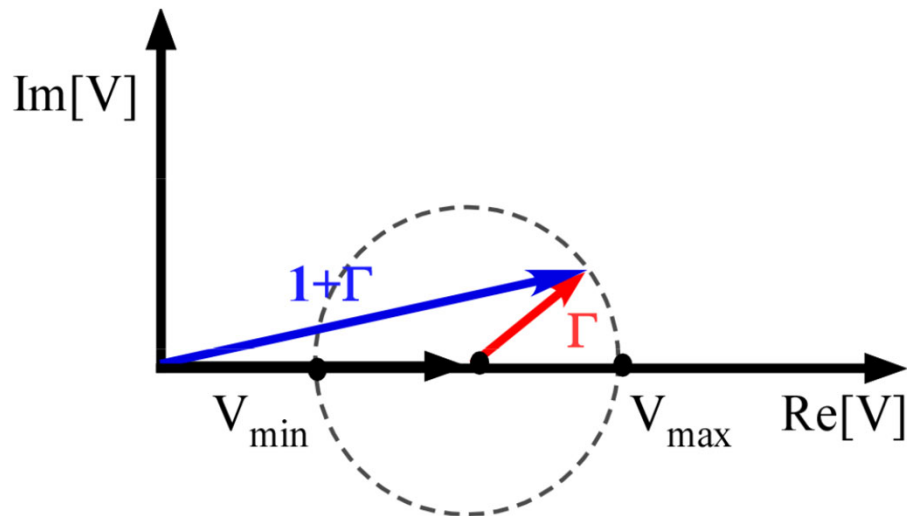
$$V_{\min} = \left[1 - |\Gamma_R| \right]$$



Voltage Standing Wave Ratio (VSWR)

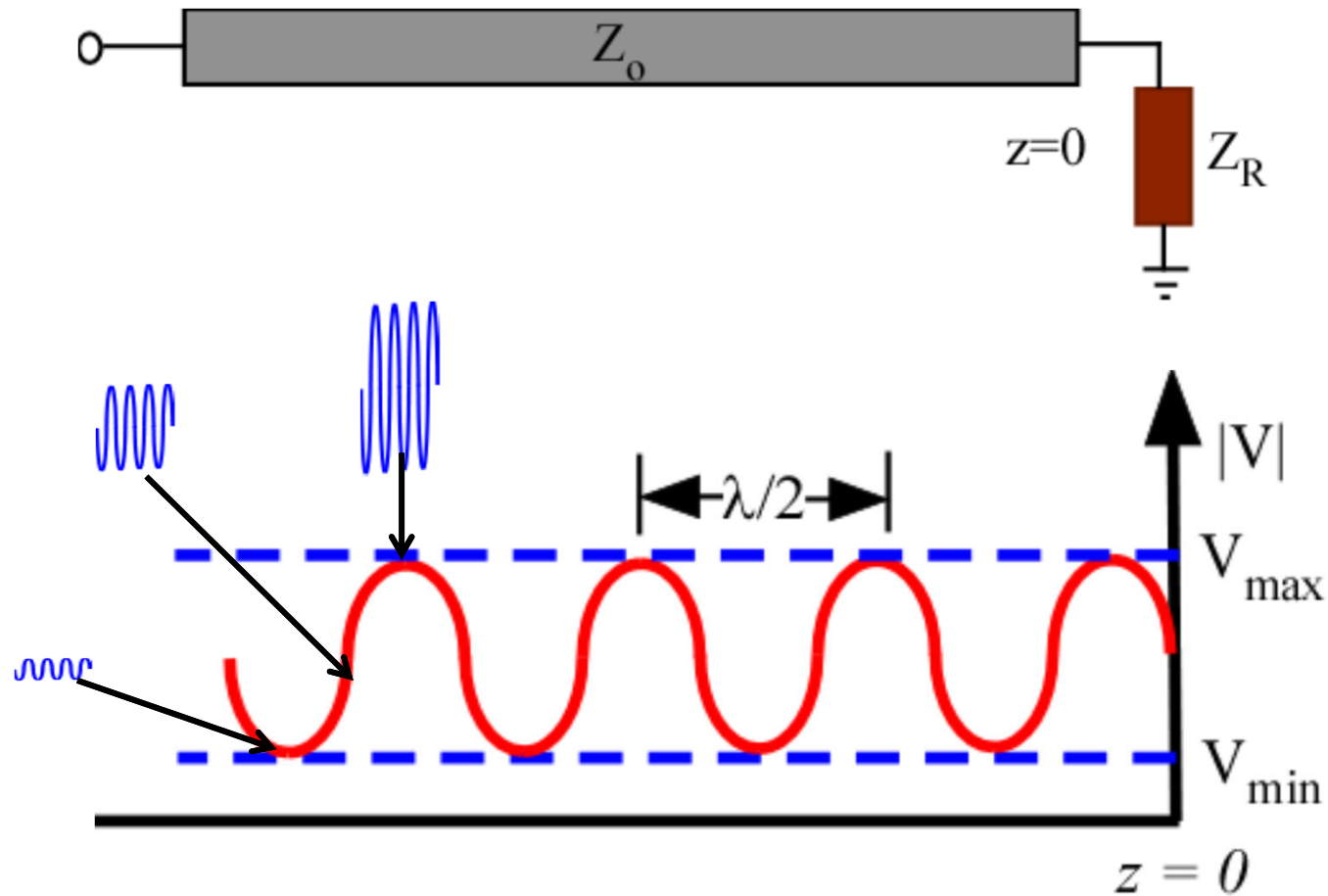
Define Voltage Standing Wave Ratio as:

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}$$



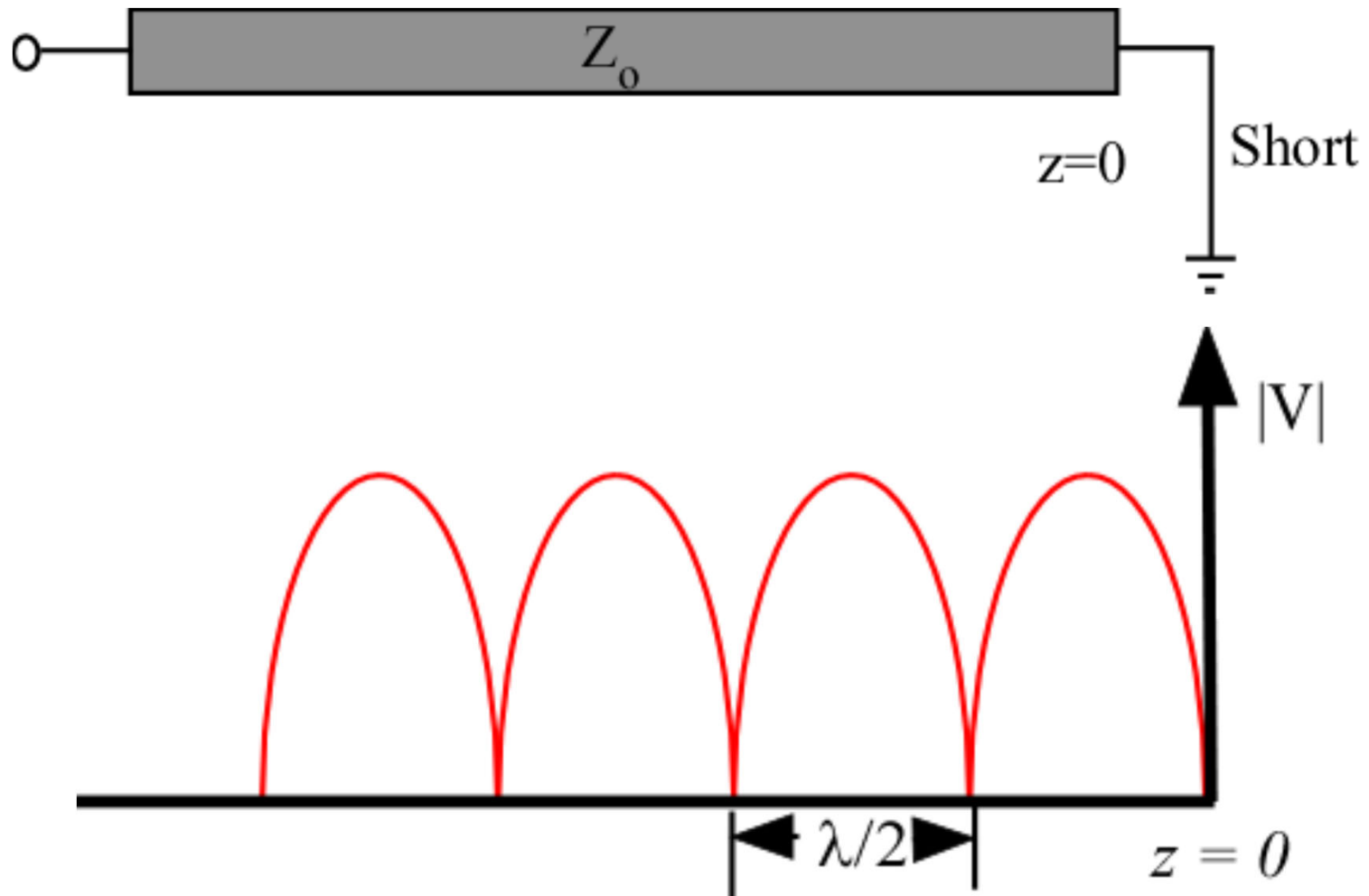
It is a measure of the interaction between forward and backward waves

VSWR – Arbitrary Load



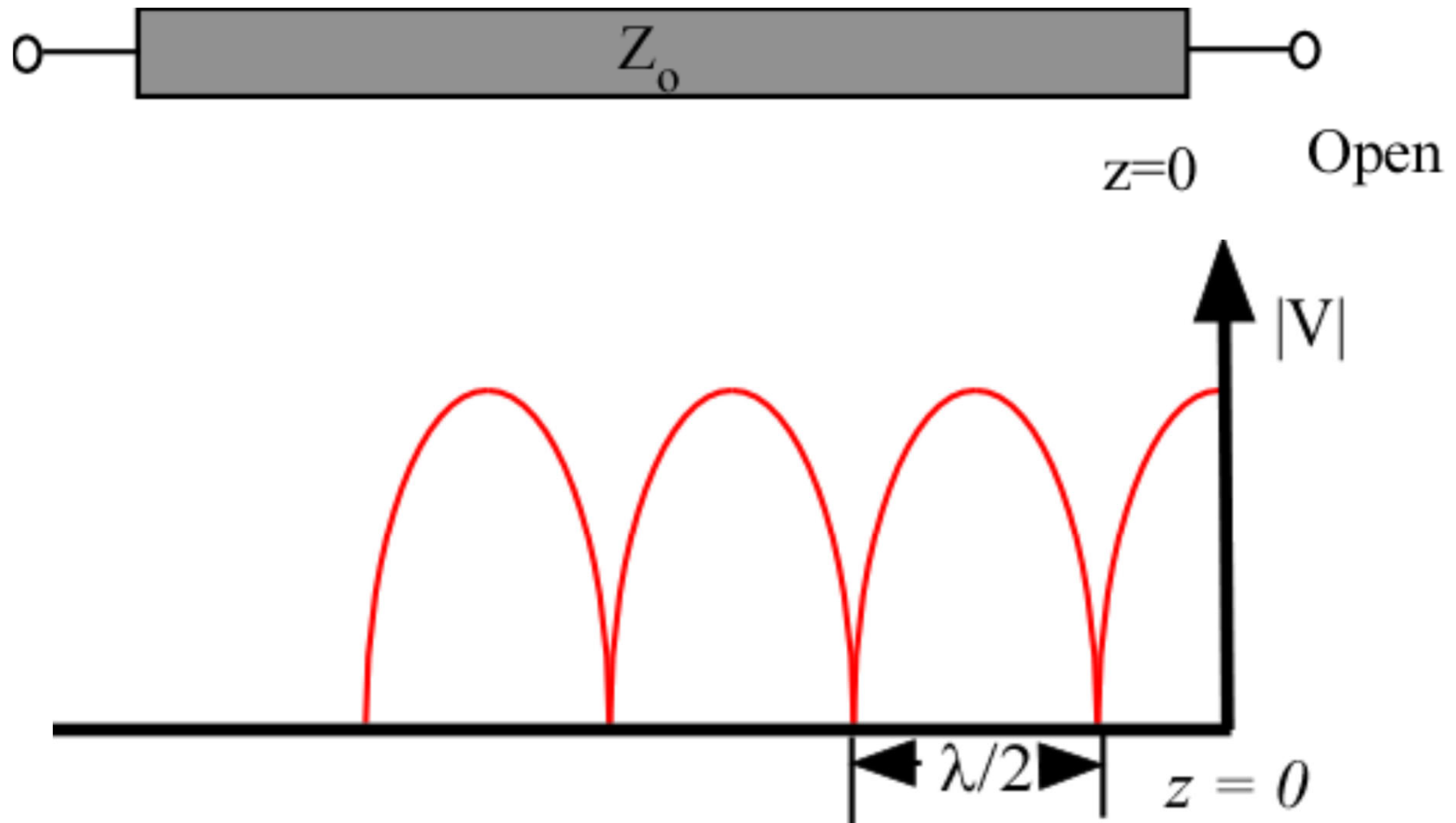
Shows variation of amplitude along line

VSWR – For Short Circuit Load



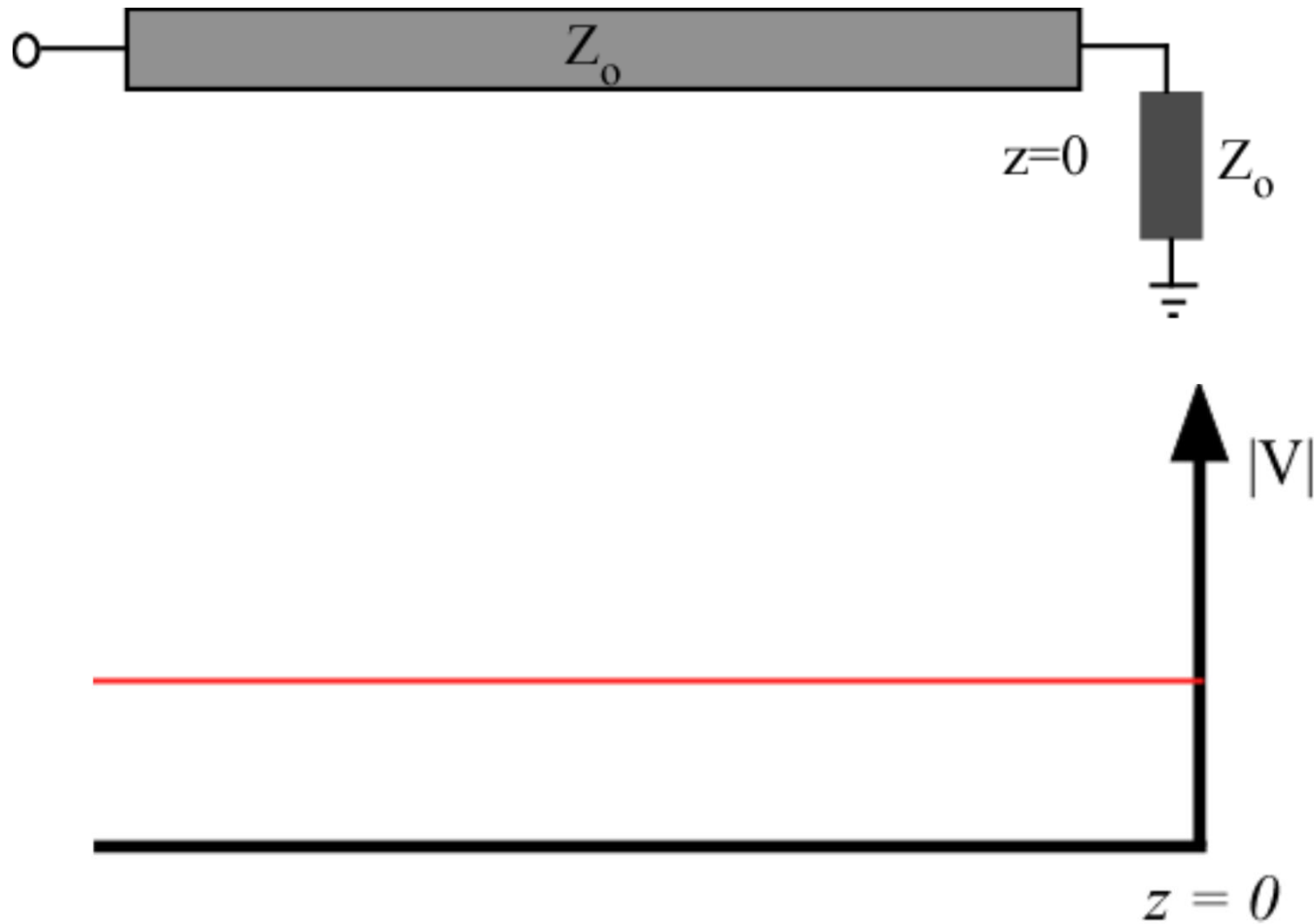
Voltage minimum is reached at load

VSWR – For Open Circuit Load



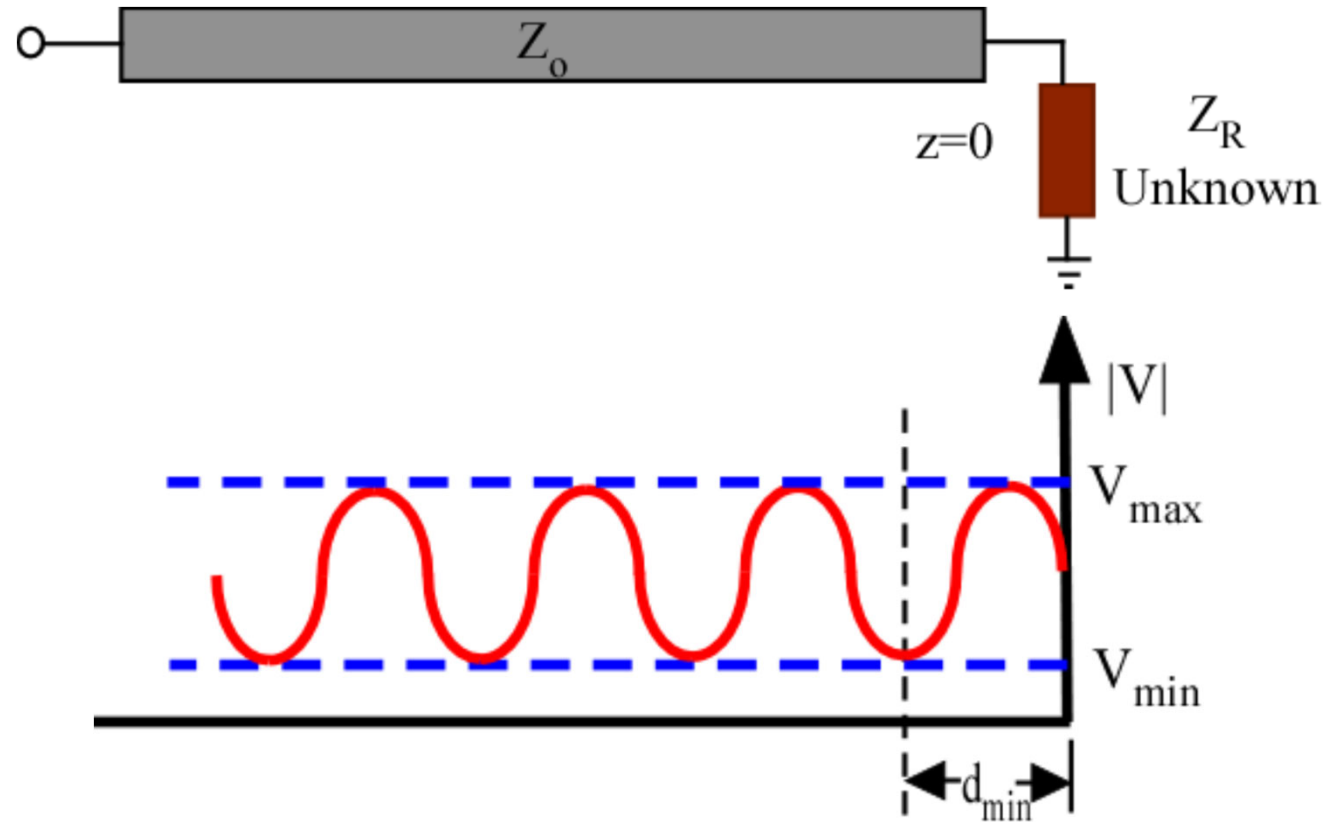
Voltage maximum is reached at load

VSWR – For Open Matched Load



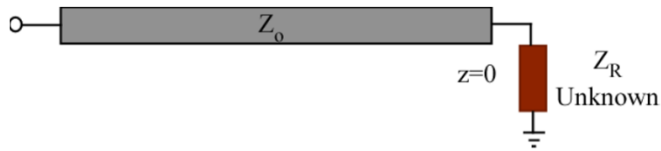
No variation in amplitude along line

Application: Slotted-Line Measurement

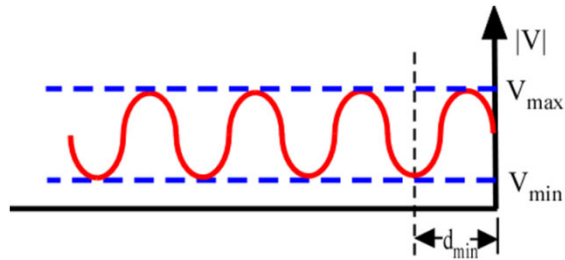


- Measure $VSWR = V_{\max}/V_{\min}$
- Measure location of first minimum

Application: Slotted-Line Measurement



At minimum, $\Gamma(z) = \text{pure real} = -|\Gamma_R|$



Therefore, $\Gamma(-d_{\min}) = \Gamma_R e^{-2j\beta d_{\min}} = -|\Gamma_R|$

So, $\Gamma_R = -|\Gamma_R| e^{+2j\beta d_{\min}}$

Since $|\Gamma_R| = \frac{VSWR - 1}{VSWR + 1}$ **then** $\Gamma_R = -\left(\frac{VSWR - 1}{VSWR + 1}\right) e^{+2j\beta d_{\min}}$

and $Z_R = Z_0 \left(\frac{1 + \Gamma_R}{1 - \Gamma_R}\right)$

Summary of TL Equations

Voltage

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

Current

$$I(z) = \frac{V_+}{Z_o} e^{-j\beta z} \left[1 - \Gamma_R e^{+2j\beta z} \right]$$

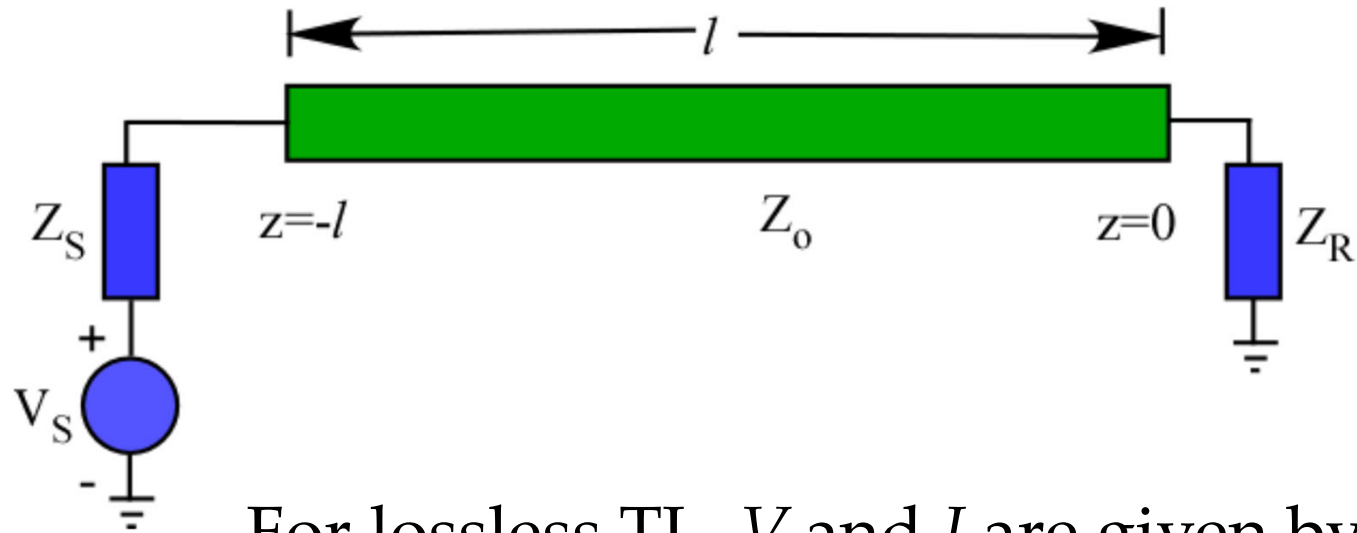
Impedance Transformation → $Z(-l) = Z_o \left[\frac{Z_R + jZ_o \tan \beta l}{Z_o + jZ_R \tan \beta l} \right]$

Reflection Coefficient Transformation → $\Gamma(-l) = \Gamma_R e^{-2j\beta l}$

Reflection Coefficient – to Impedance → $Z(z) = Z_o \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$

Impedance to Reflection Coefficient → $\Gamma(z) = \frac{Z(z) - Z_o}{Z(z) + Z_o}$

Determining V_+



For lossless TL, V and I are given by

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

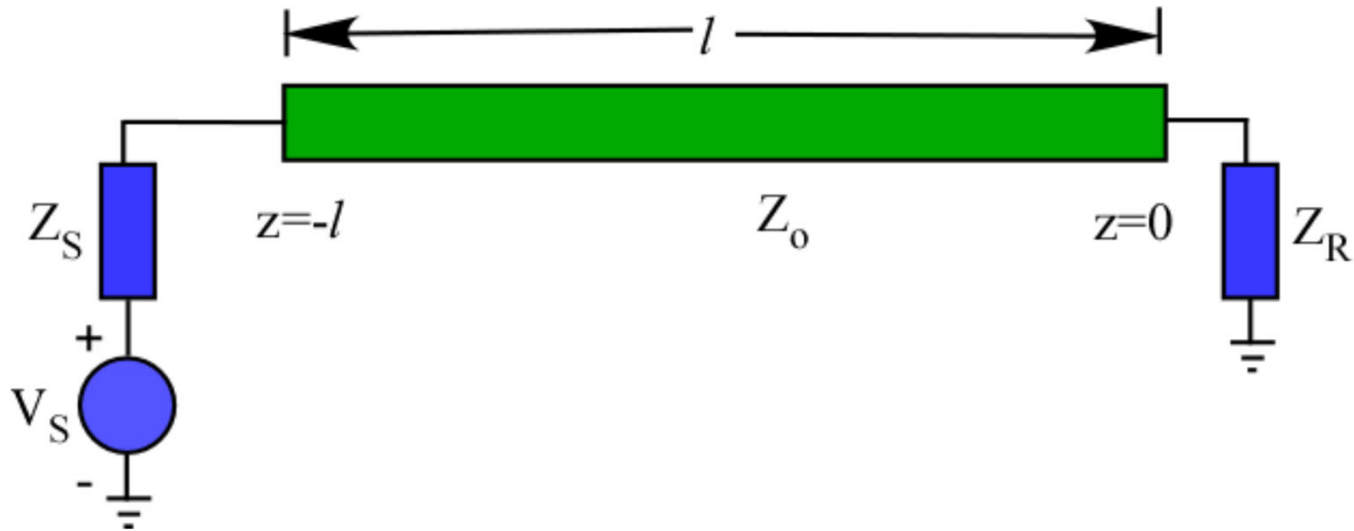
reflection coefficient
at the load

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$$

$$I(z) = \frac{V_+ e^{-j\beta z}}{Z_o} \left[1 - \Gamma_R e^{+2j\beta z} \right]$$

$$\text{At } z = -l, \quad V_S = Z_S I(-l) + V(-l)$$

Determining V_+



this leads to

$$V_S = V_+ e^{+j\beta l} (1 + \Gamma_R e^{-2j\beta l}) + \frac{Z_S}{Z_o} V_+ e^{+j\beta l} (1 - \Gamma_R e^{-2j\beta l})$$

or

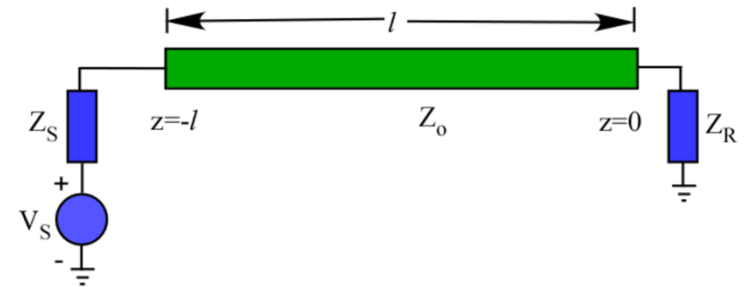
$$V_S = V_+ \left(e^{+j\beta l} + \Gamma_R e^{-j\beta l} + \frac{Z_S}{Z_o} e^{+j\beta l} - \Gamma_R \frac{Z_S}{Z_o} e^{-j\beta l} \right)$$

$$V_S = V_+ \left(e^{+j\beta l} \left(1 + \frac{Z_S}{Z_o} \right) + \Gamma_R e^{-j\beta l} \left(1 - \frac{Z_S}{Z_o} \right) \right)$$

Determining V_+

Divide through by $\left(1 + \frac{Z_S}{Z_o}\right) = \frac{1}{T_S}$

$$V_+ \left(e^{+j\beta l} - \Gamma_S \Gamma_R e^{-j\beta l} \right) = T_S V_S$$

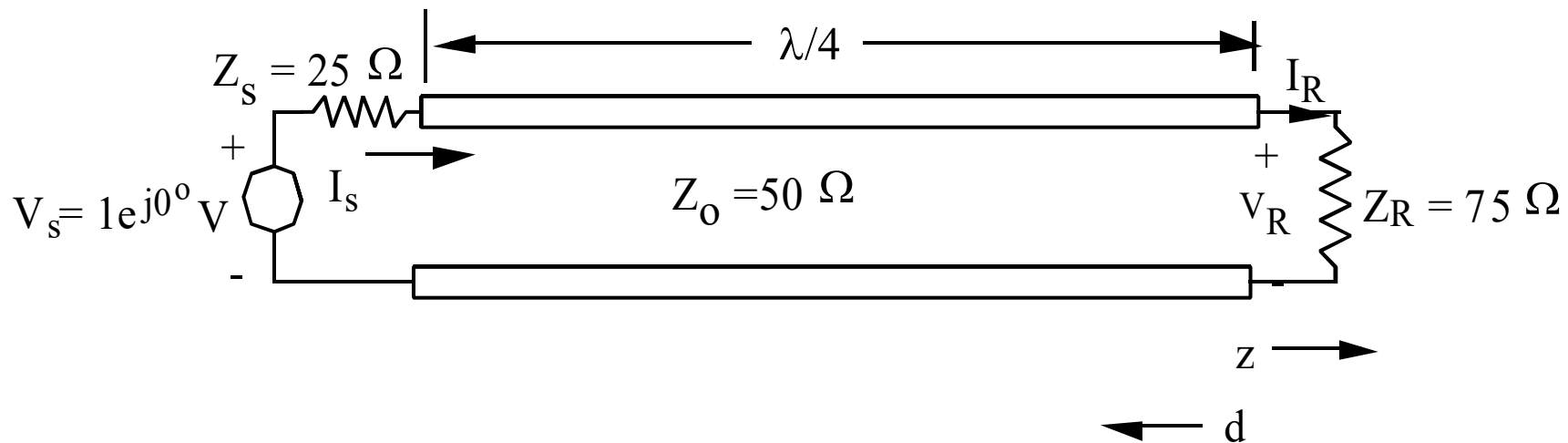


$$\text{with } T_S = \left(1 + \frac{Z_S}{Z_o}\right)^{-1} = \frac{Z_o}{Z_S + Z_o} \quad \text{and} \quad \Gamma_S = \frac{Z_S - Z_o}{Z_S + Z_o}$$

From which

$$V_+ = \frac{T_S V_S e^{-j\beta l}}{1 - \Gamma_S \Gamma_R e^{-2j\beta l}}$$

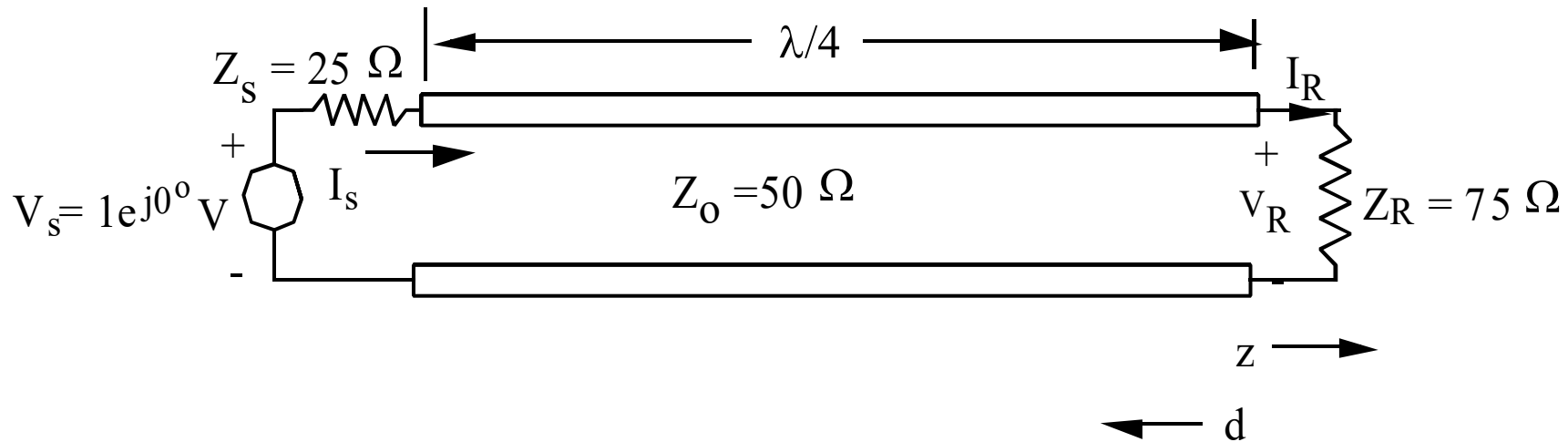
TL Example



A signal generator having an internal resistance $Z_s = 25 \Omega$ and an open circuit phasor voltage $V_s = 1e^{j0}$ volt is connected to a $50\text{-}\Omega$ lossless transmission line as shown in the above picture. The load impedance is $Z_R = 75 \Omega$ and the line length is $\lambda/4$.

Find the magnitude and phase of the load current I_R .

TL Example – Cont'



$$V_+ = \frac{T_s V_s e^{-j\beta l}}{1 - \Gamma_R \Gamma_s e^{-2j\beta l}}$$

$$T_s = \frac{Z_o}{Z_s + Z_o} = \frac{50}{50 + 25} = 2/3$$

$$\Gamma_s = \frac{Z_s - Z_o}{Z_s + Z_o} = \frac{25 - 50}{25 + 50} = -1/3$$

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o} = \frac{75 - 50}{75 + 50} = 1/5$$

TL Example – Cont'

$$\beta l = \frac{2\pi}{\lambda} \frac{\lambda}{4} = \frac{\pi}{2} \Rightarrow e^{-j\beta l} = -j$$

$$V_+ = \frac{(2/3)(1)(-j)}{1 - (-1/3)(1/5)(-1)} = \frac{-j2/3}{1 - 1/15} = -j5/7$$

$$V_+ = -j0.714285 \text{ V}$$

$$I_R = \frac{V_+}{Z_o} [1 - \Gamma_R] = -j \frac{0.714285}{50} [1 - 0.2] = -j \frac{0.714285 \times 0.8}{50}$$

$$I_R = -j0.0114285 \text{ A}$$

Frequency-Domain TL Solution

$$V(z) = V_+ e^{-j\beta z} \left[1 + \Gamma_R e^{+2j\beta z} \right]$$

$$I(z) = \frac{V_+ e^{-j\beta z}}{Z_o} \left[1 - \Gamma_R e^{+2j\beta z} \right]$$

$$V_+ = \frac{T_S V_S e^{-j\beta l}}{1 - \Gamma_S \Gamma_R e^{-2j\beta l}}$$

Magnitude of reflection coefficients at source and load is less than 1.

Geometric Series Expansion

$$\text{Since } \left| \Gamma_S \Gamma_R e^{-2j\beta l} \right| \leq 1$$

V_+ can be expanded in a geometric series form

$$V_+ = \frac{T_S V_S e^{-j\beta l}}{1 - \Gamma_S \Gamma_R e^{-2j\beta l}} \quad V_+ = T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^k e^{-2j\beta k l} e^{-j\beta l}$$

$$V(z) = T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^k e^{-j\beta l(2k+1)} e^{-j\beta z} + T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^{k+1} e^{-j\beta l(2k+1)} e^{+j\beta z}$$

$$\beta = \frac{\omega}{v}$$

$$V(z) = T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^k e^{-j\frac{\omega}{v}[z+(2k+1)l]} + T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^{k+1} e^{-j\frac{\omega}{v}[z+(2k+1)l]}$$

TL Time-Domain Solution

$$v(z,t) = T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^k v_s \left(t - \frac{(2k+1)l + z}{v_o} \right) \\ + T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^{k+1} v_s \left(t - \frac{(2k+1)l - z}{v_o} \right)$$

at $z=0$

$$v(0,t) = T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^k v_s \left(t - \frac{(2k+1)l}{v_o} \right) \\ + T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^{k+1} v_s \left(t - \frac{(2k+1)l}{v_o} \right)$$

TL Time-Domain Solution

At $z=-l$

$$v(-l, t) = T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^k v_s \left(t - \frac{(2k+1)l + l}{v_o} \right)$$

$$+ T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^{k+1} v_s \left(t - \frac{(2k+1)l + l}{v_o} \right)$$

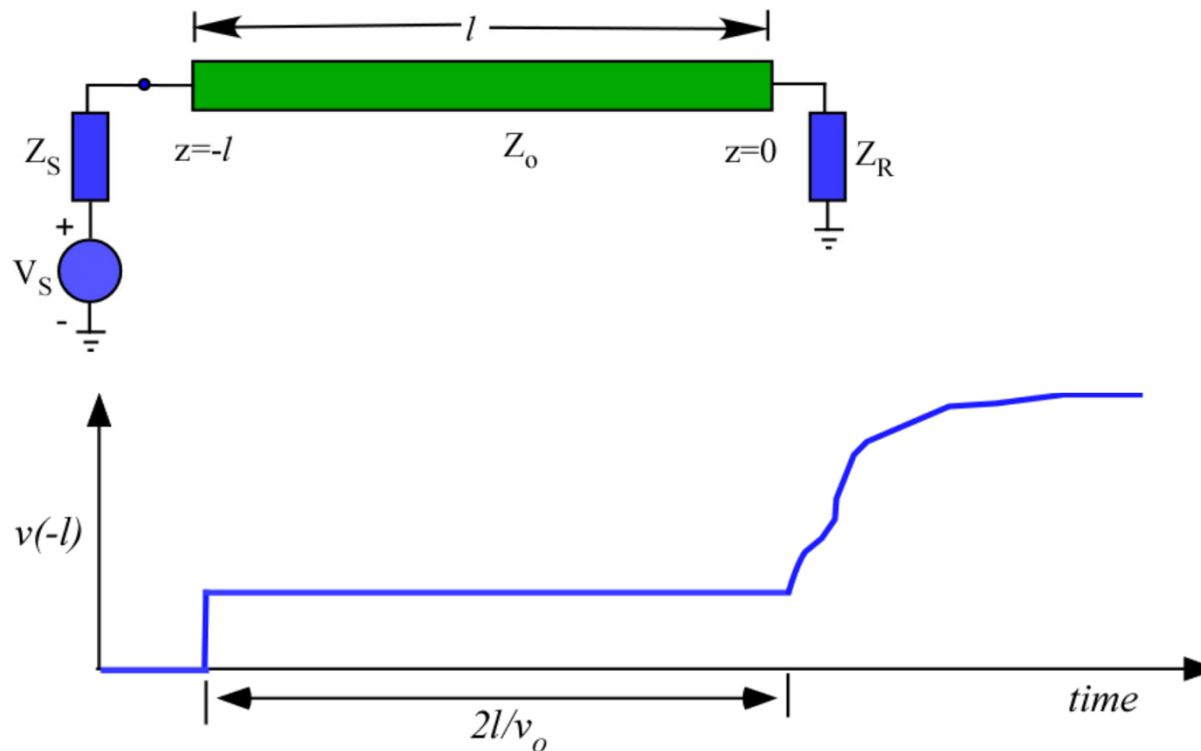
$$v(-l, t) = T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^k v_s \left(t - \frac{2kl}{v_o} \right)$$

$$+ T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^{k+1} v_s \left(t - \frac{2(k+1)l}{v_o} \right)$$

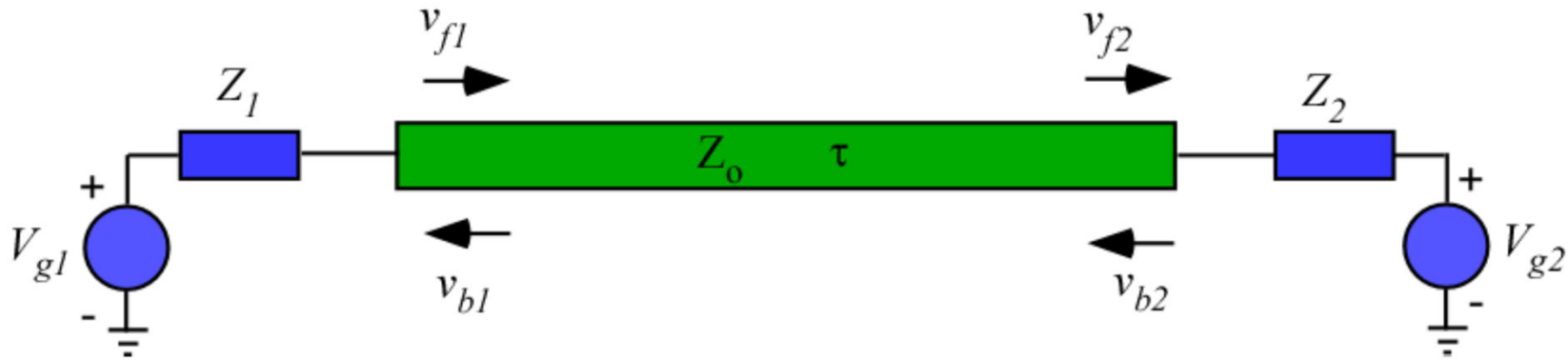
TL - Time-Domain Reflectometer

For TDR, $Z_S = Z_0 \rightarrow \Gamma_S = 0$, and retain only $k=1$

$$v(-l, t) = T_S v_s(t) + T_S \Gamma_R v_s\left(t - \frac{2l}{v_o}\right)$$



Wave Shifting Method



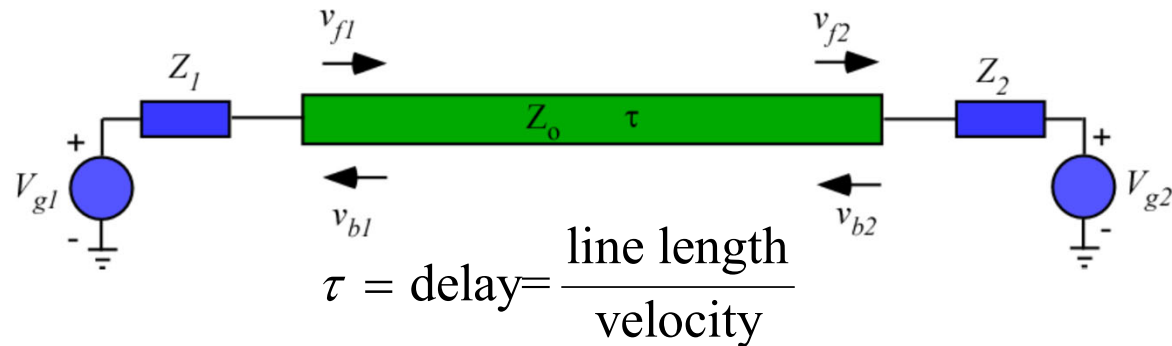
$v_{f1}(t)$ Forward traveling wave at port 1 (measured at near end of line)

$v_{f2}(t)$ Forward traveling wave at port 2 (measured at far end of line)

$v_{b1}(t)$ Backward traveling wave at port 1 (measured at near end of line)

$v_{b2}(t)$ Backward traveling wave at port 2 (measured at far end of line)

Wave Shifting Solution*



$$v_{f1}(t) = \Gamma_1 v_{b2}(t - \tau) + T_1 V_{g1}(t)$$

$$v_{b2}(t) = \Gamma_2 v_{f1}(t - \tau) + T_2 V_{g2}(t)$$

$$v_{f2}(t) = v_{f1}(t - \tau)$$

$$v_{b1}(t) = v_{b2}(t - \tau)$$

$$v_1(t) = v_{f1}(t) + v_{b1}(t)$$

$$v_2(t) = v_{f2}(t) + v_{b2}(t)$$

$$\Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o}$$

$$T_1 = \frac{Z_o}{Z_1 + Z_o}$$

$$\Gamma_2 = \frac{Z_2 - Z_o}{Z_2 + Z_o}$$

$$T_2 = \frac{Z_o}{Z_2 + Z_o}$$

*Schutt-Aine & Mittra, *Trans. Microwave Theory Tech.*, pp. 529-536, vol. 36 March 1988.

Frequency Dependence of Lumped Circuit Models

- At higher frequencies, a lumped circuit model is no longer accurate for interconnects and one must use a distributed model
- Transition frequency depends on the dimensions and relative magnitude of the interconnect parameters.

$$f \approx \frac{0.3 \times 10^9}{10d\sqrt{\epsilon_r}}$$

$$t_r \approx \frac{0.35}{f}$$

Lumped Circuit or Transmission Line?

- **Determine frequency or bandwidth of signal**
 - RF/Microwave: $f = \text{operating frequency}$
 - Digital: $f = 0.35/t_r$
- **Determine the propagation velocity and wavelength**
 - Material medium $v = c/(\epsilon_r)^{1/2}$
 - Obtain wavelength $\lambda = v/f$
- **Compare wavelength with feature size**
 - If $\lambda \gg d$, use lumped circuit: $L_{\text{tot}} = L * \text{length}$, $C_{\text{tot}} = C * \text{length}$
 - If $\lambda \approx 10d$ or $\lambda < 10d$, use transmission-line model

Frequency Dependence of Lumped Circuit Models

<u>Level</u>	<u>Dimension</u>	<u>Frequency</u>	<u>Edge rate</u>
PCB line	10 in	> 55 MHz	< 7ns
Package	1 in	> 400 MHz	< 0.9 ns
VLSI int*	100 um	> 8 GHz	< 50 ps

* Using RC criterion for distributed effect