ECE 546
Lecture - 07
Multiconductors

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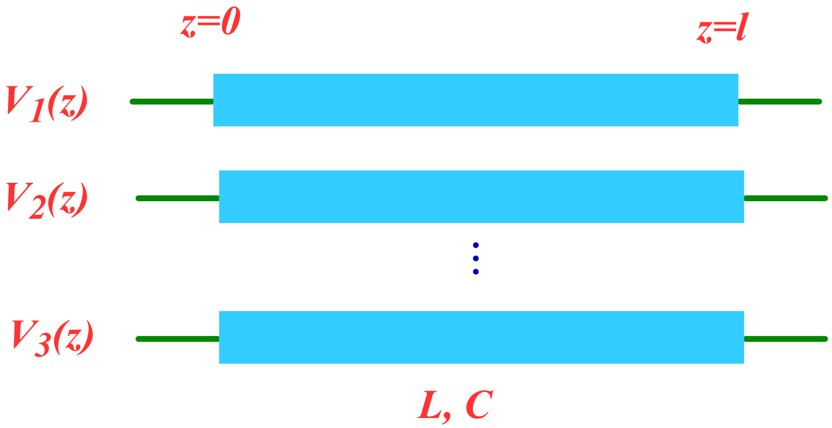
TELGRAPHER’S EQUATION FOR N COUPLED TRANSMISSION LINES



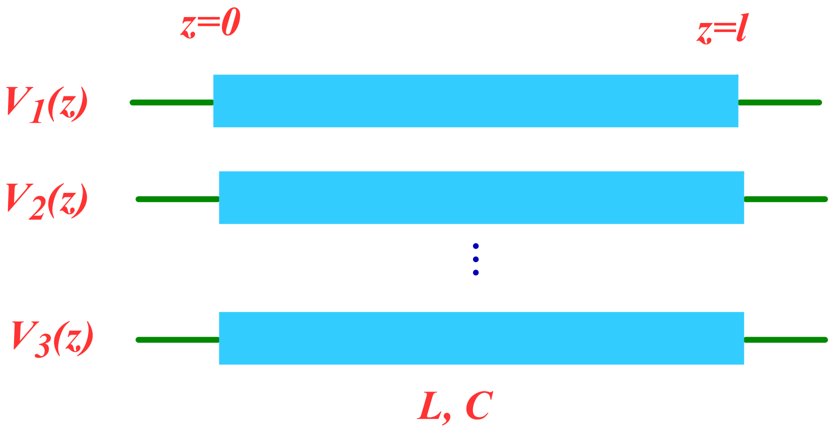
$$-\frac{\partial V}{\partial z} = L\frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial z} = C\frac{\partial V}{\partial t}$$

V and *I* are the line voltage and line current VECTORS respectively (dimension n).

N-LINE SYSTEM

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ V_n \end{bmatrix}$$



$$I = \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ I_n \end{bmatrix}$$

L and *C* are the inductance and capacitance MATRICES respectively

$$L = \begin{bmatrix} L_{11} & L_{12} & \cdot & \cdot \\ L_{21} & L_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & L_{nn} \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot \\ C_{21} & C_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}$$

N-LINE ANALYSIS

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = CL\frac{\partial^2 I}{\partial t^2}$$

In general $LC \neq CL$ and LC and CL are not symmetric matrices.

GOAL: Diagonalize LC and CL which will result in a transformation on the variables V and I.

Diagonalize LC and CL is equivalent to finding the eigenvalues of LC and CL.

Since LC and CL are adjoint, they must have the same eigenvalues.

MODAL ANALYSIS

$$\frac{\partial^2 EV}{\partial z^2} = ELCE^{-1} \frac{\partial^2 EV}{\partial t^2}$$

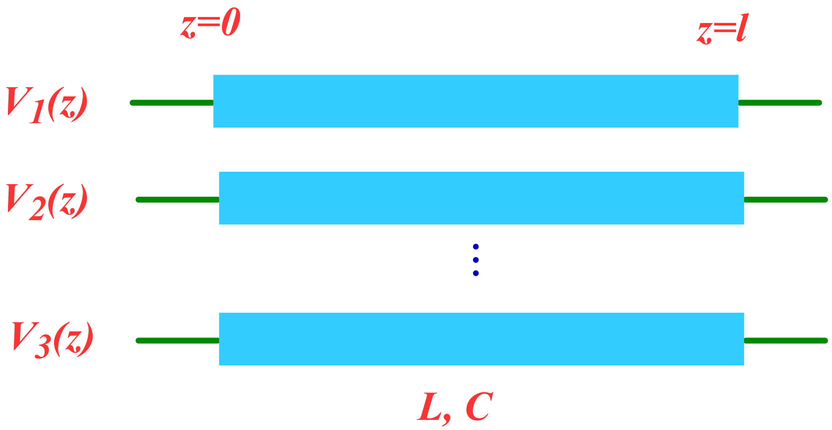
$$\frac{\partial^2 HI}{\partial z^2} = HCLH^{-1} \frac{\partial^2 HI}{\partial t^2}$$

LC and CL are adjoint matrices. Find matrices E and H such that

$$ELCE^{-1} = HCLH^{-1} = \Lambda_m^2$$

is the diagonal eigenvalue matrix whose entries are the inverses of the modal velocities squared.

MODAL ANALYSIS



$$\frac{\partial^2 I_m}{\partial z^2} = \Lambda_m^2 \frac{\partial^2 I_m}{\partial t^2}$$

$$\frac{\partial^2 V_m}{\partial z^2} = \Lambda_m^2 \frac{\partial^2 V_m}{\partial t^2}$$

$$V_m = EV \qquad I_m = HI$$

Second-order differential equation in modal space

EIGENVECTORS

$$V_m = EV$$

$$I_m = HI$$

V_m and I_m are the modal voltage and current vectors respectively.

$$V_m = \begin{bmatrix} V_{m1} \\ V_{m2} \\ \cdot \\ \cdot \\ V_{mn} \end{bmatrix} \qquad I_m = \begin{bmatrix} I_{m1} \\ I_{m2} \\ \cdot \\ \cdot \\ I_{mn} \end{bmatrix}$$

EIGEN ANALYSIS

- **A scalar λ is an eigenvalue of a matrix A if there exists a vector X such that $AX = \lambda X$. (i.e. for which multiplication by a matrix is equivalent to an elongation of the vector).**

- **The vector X which satisfies the above requirement is an eigenvector of A.**

- **The eigenvalues λ of A satisfy the relation $|A - \lambda I| = 0$ where I is the unit matrix.**

EIGEN ANALYSIS

Assume A is an n x n matrix with n distinct eigenvalues, then,

- **A has n linearly independent eigenvectors.**
- **The n eigenvectors can be arranged into an n x n matrix E; the eigenvector matrix.**
- **Finding the eigenvalues of A is equivalent to diagonalizing A.**
- **Diagonalization is achieved by finding the eigenvector matrix E such that EAE^{-1} is a diagonal matrix.**

EIGEN ANALYSIS

For an *n*-line system, it can be shown that

- * **LC can be transformed into a diagonal matrix whose entries are the eigenvalues.**
- * **LC possesses *n* distinct eigenvalues (possibly degenerate).**
- * **There exist *n* eigenvectors which are linearly independent.**
- * **Each eigenvalue is associated with a mode; the propagation velocity of that mode is the inverse of the eigenvalue.**

EIGEN ANALYSIS

* Each eigenvector is associated to an eigenvalue and therefore to a particular mode.

* Each normalized eigenvector represents the relative line excitation required to excite the associated mode.

EIGENVECTORS

E : Voltage eigenvector matrix

$$ELCE^{-1} = \Lambda_m^2$$

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

H : Current eigenvector matrix

$$HCLH^{-1} = \Lambda_m^2$$

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Eigenvalues and Eigenvectors

$$ELCE^{-1} = \Lambda_m^2$$

gives

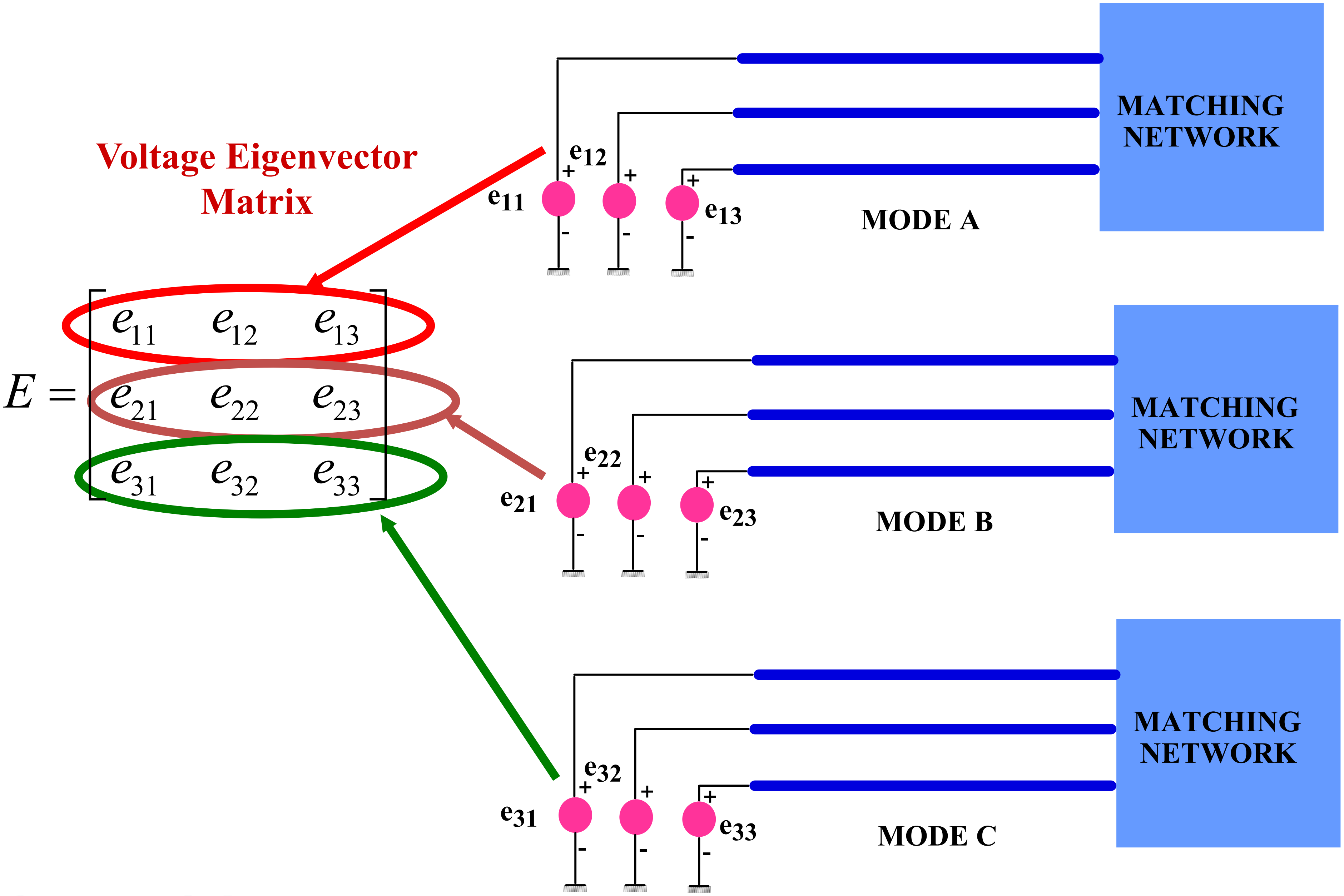
$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \qquad \Lambda_m = \begin{bmatrix} \dfrac{1}{v_{m1}} & 0 & 0 \\ 0 & \dfrac{1}{v_{m2}} & 0 \\ 0 & 0 & \dfrac{1}{v_{m3}} \end{bmatrix}$$

$$HCLH^{-1} = \Lambda_m^2$$

gives

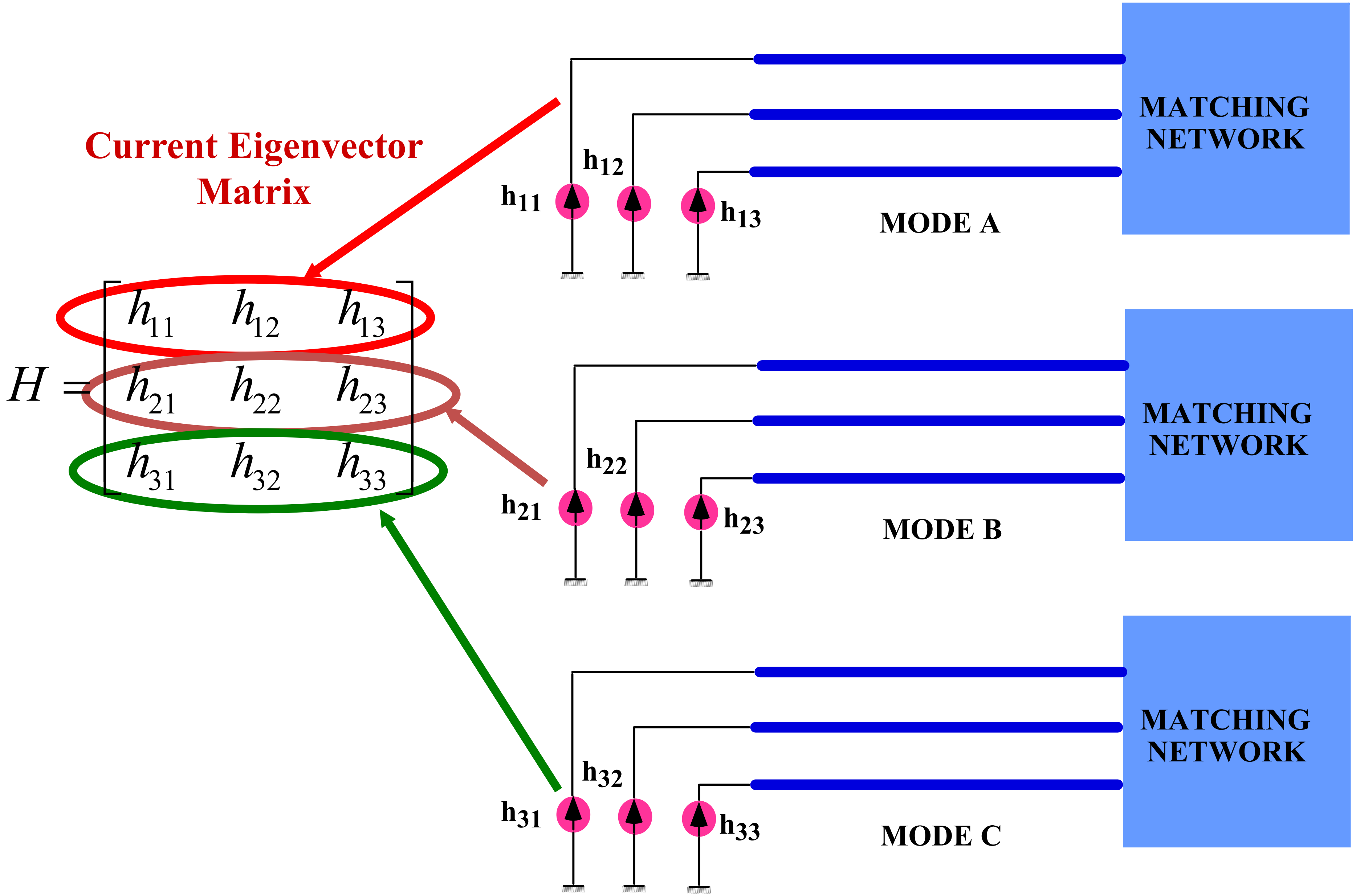
$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \qquad \Lambda_m = \begin{bmatrix} \dfrac{1}{v_{m1}} & 0 & 0 \\ 0 & \dfrac{1}{v_{m2}} & 0 \\ 0 & 0 & \dfrac{1}{v_{m3}} \end{bmatrix}$$

Modal Voltage Excitation



$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

Modal Current Excitation



Current Eigenvector Matrix

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

EIGENVECTORS

E **: voltage eigenvector matrix**

H **: current eigenvector matrix**

E **and** ***H*** **depend on both** ***L*** **and** ***C***

Special case: For a two-line *symmetric* system, E and H are equal and independent of L and C. The eigenvectors are: [1,1] and [1,-1]

Line variables are recovered using

$$V = E^{-1} V_m$$

$$I = H^{-1} I_m$$

MODAL AND LINE IMPEDANCE MATRICES

By requiring $V_m = Z_m I_m$, we arrive at

$$Z_m = \Lambda_m^{-1} E L H^{-1}$$

Z_m is a diagonal impedance matrix which relates modal voltages to modal currents

By requiring $V = Z_c I$, one can define a line impedance matrix Z_c. Z_c relates line voltages to line currents

Z_m is diagonal. Z_c contains nonzero off-diagonal elements. We can show that

$$Z_c = E^{-1} Z_m H = E^{-1} \Lambda_m^{-1} E L$$

EIGENVALUE AND PROPAGATION MATRICES

Eigenvalue matrix

$$\Lambda_m = \begin{bmatrix} \frac{1}{v_{m1}} & & & \\ & \frac{1}{v_{m2}} & & \\ & & \bullet & \\ & & & \frac{1}{v_{mn}} \end{bmatrix}$$

Propagation matrix

$$W(u) = \begin{bmatrix} e^{-\frac{j\omega u}{v_{m1}}} & & & \\ & e^{-\frac{j\omega u}{v_{m2}}} & & \\ & & \bullet & \\ & & & e^{-\frac{j\omega u}{v_{mn}}} \end{bmatrix}$$

MATRIX CONSTRUCTION

$$[L]_{ii} = L_s \qquad [L]_{ij} = L_{ij}^{(m)}$$

$$[C]_{ii} = C_s + \sum_{j=1}^{n} C_{ij}^{(m)} \qquad [C]_{ij} = -C_{ij}^{(m)}$$

$$[R]_{ii} = R_s \qquad [R]_{ij} = R_{ij}^{m}$$

$$[G]_{ii} = G_s + \sum_{j=1}^{n} G_{ij}^{(m)} \qquad [\mathbf{G}]_{ij} = -G_{ij}^{(m)}$$

Modal Variables

$$V_m = EV \qquad\qquad I_m = HI$$

General solution in modal space is:

$$V_m(z) = W(z)A + W(-z)B$$

$$I_m(z) = Z_m^{-1}\left[W(z)A - W(-z)B\right]$$

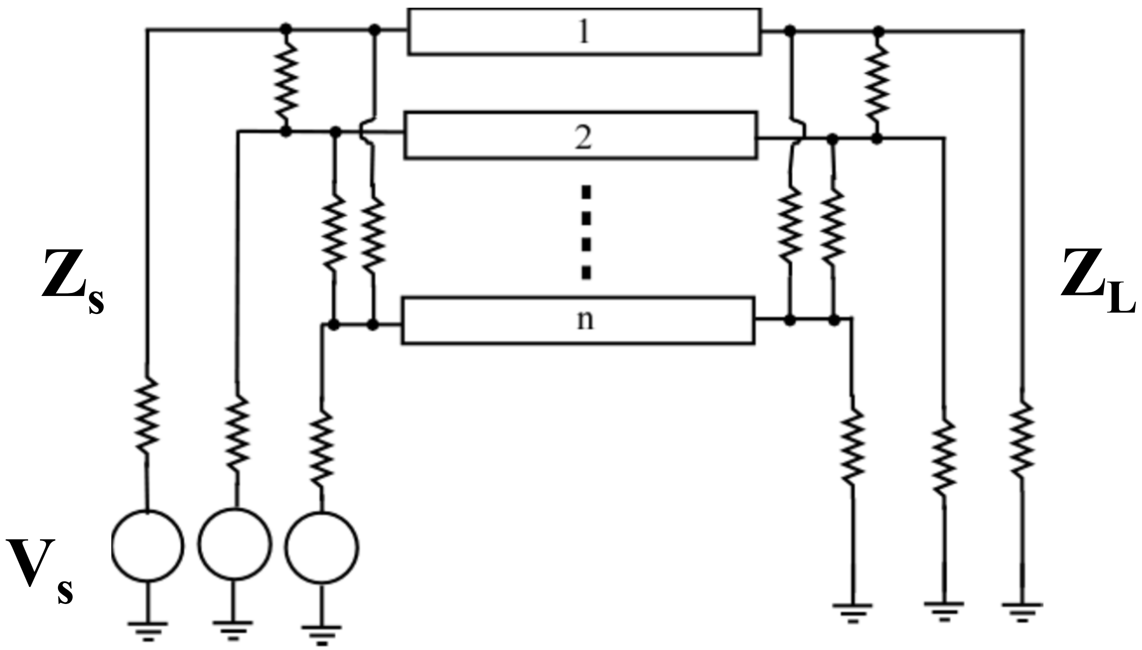
Modal impedance matrix

$$Z_m = \Lambda_m^{-1} E L H^{-1}$$

Line impedance matrix

$$Z_c = E^{-1} Z_m H = E^{-1} \Lambda_m^{-1} E L$$

N-Line Network



Z_s : Source impedance matrix

Z_L : Load impedance matrix

V_s : Source vector

Reflection Coefficients

Source reflection coefficient matrix

$$\Gamma_S = -\left[1_n + EZ_S L^{-1} E^{-1} \Lambda_m\right]^{-1} \left[1_n - EZ_S L^{-1} E^{-1} \Lambda_m\right]$$

Load reflection coefficient matrix

$$\Gamma_L = -\left[1_n + EZ_L L^{-1} E^{-1} \Lambda_m\right]^{-1} \left[1_n - EZ_L L^{-1} E^{-1} \Lambda_m\right]$$

- **The reflection and transmission coefficient are *n x n* matrices with off-diagonal elements. The off-diagonal elements account for the coupling between modes.**

- **Z_s and Z_L can be chosen so that the reflection coefficient matrices are zero**

Matching Network for Multiconductors

Load reflection coefficient matrix

$$\Gamma_L = -\left[1_n + EZ_L L^{-1} E^{-1} \Lambda_m\right]^{-1} \left[1_n - EZ_L L^{-1} E^{-1} \Lambda_m\right]$$

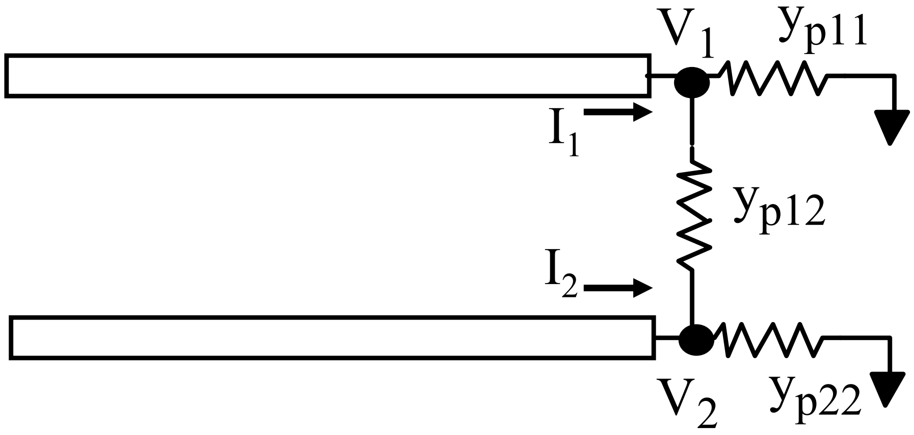
If $Z_L = Z_c = E^{-1} \Lambda_m^{-1} EL$

then $\Gamma_L = -\left[1_n + EE^{-1} \Lambda_m^{-1} ELL^{-1} E^{-1} \Lambda_m\right]^{-1} \left[1_n - EE^{-1} \Lambda_m^{-1} ELL^{-1} E^{-1} \Lambda_m\right]$

and $\Gamma_L = 0$

***** ***Z***

TERMINATION NETWORK



$$I_1 = y_{p11}V_1 + y_{p12}\left(V_1 - V_2\right)$$

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$y_{11} = y_{p11} + y_{p12}$$

$$I_2 = y_{p22}V_2 + y_{p12}\left(V_2 - V_1\right)$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

$$y_{22} = y_{p22} + y_{p12}$$

$$y_{12} = y_{21} = -y_{p12}$$

In general for a multiline system

$$\mathbf{I} = \mathbf{YV} \Rightarrow \mathbf{V} = \mathbf{ZY} \quad \mathbf{Z} = \mathbf{Y}^{-1}$$

Note: $y_{ii} \neq y_{pii}$

$$y_{ij} = -y_{pij} \quad \text{for } i \neq j$$

$$z_{ij} \neq \frac{1}{y_{pij}}$$

Termination Network Construction

To get impedance matrix Z

- Get physical impedance values y_{pij}
- Calculate y_{ij}'s from y_{pij}'s
- Construct Y matrix
- Invert Y matrix to obtain Z matrix

Remark: If $y_{pij} = 0$ for all $i \neq j$, then $Y = Z^{-1}$ and $z_{ii} = 1/y_{ii}$

Procedure for Multiconductor Solution

1) Get L and C matrices and calculate LC product

2) Get square root of eigenvalues and eigenvectors of LC matrix → Λ_m

3) Arrange eigenvectors into the voltage eigenvector matrix E

4) Get square root of eigenvalues and eigenvectors of CL matrix →Λ_m

5) Arrange eigenvectors into the current eigenvector matrix H

6) Invert matrices E, H, Λ_m.

7) Calculate the line impedance matrix $Z_c \rightarrow Z_c = E^{-1}\Lambda_m^{-1}EL$

Procedure for Multiconductor Solution

8) Construct source and load impedance matrices $Z_s(t)$ and $Z_L(t)$

9) Construct source and load reflection coefficient matrices $\Gamma_1(t)$ and $\Gamma_2(t)$.

10) Construct source and load transmission coefficient matrices $T_1(t)$, $T_2(t)$

11) Calculate modal voltage sources $g_1(t)$ and $g_2(t)$

12) Calculate modal voltage waves:

$$a_1(t) = T_1(t)g_1(t) + \Gamma_1(t)a_2(t - \tau m)$$

$$a_2(t) = T_2(t)g_2(t) + \Gamma_2(t)a_1(t - \tau m)$$

$$b_1(t) = a_2(t - \tau_m)$$

$$b_2(t) = a_1(t - \tau_m)$$

Wave-Shifting Solution*

$$a_i(t-\tau_m)=\begin{bmatrix} a_{i-\mathrm{mod}e-1}(t-\tau_{m1}) \\ a_{i-\mathrm{mod}e-2}(t-\tau_{m2}) \\ \bullet \\ a_{i-\mathrm{mod}e-n}(t-\tau_{mn}) \end{bmatrix}$$

τ_{mi} is the delay associated with mode i. τ_{mi} = length/velocity of mode i. The modal volage wave vectors $a_1(t)$ and $a_2(t)$ need to be stored since they contain the history of the system.

13) Calculate total modal voltage vectors:

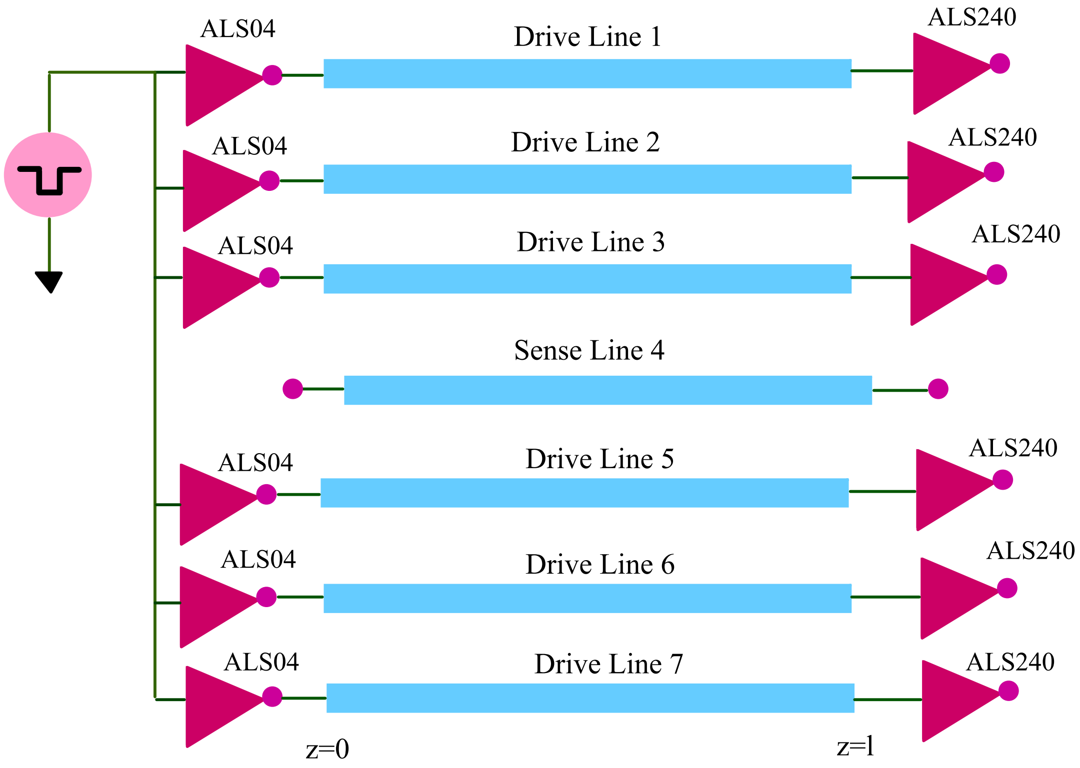
$$V_{m1}(t) = a_1(t) + b_1(t)$$
$$V_{m2}(t) = a_2(t) + b_2(t)$$

14) Calculate line voltage vectors:

$$V_1(t) = E^{-1}V_{m1}(t)$$
$$V_2(t) = E^{-1}V_{m2}(t)$$

* J. E. Schutt-Aine and R. Mittra, "Transient analysis of coupled lossy transmission lines with nonlinear terminations," IEEE Trans. Circuit Syst., vol. CAS-36, pp. 959-967, July 1989.

7-Line Coupled-Microstrip System

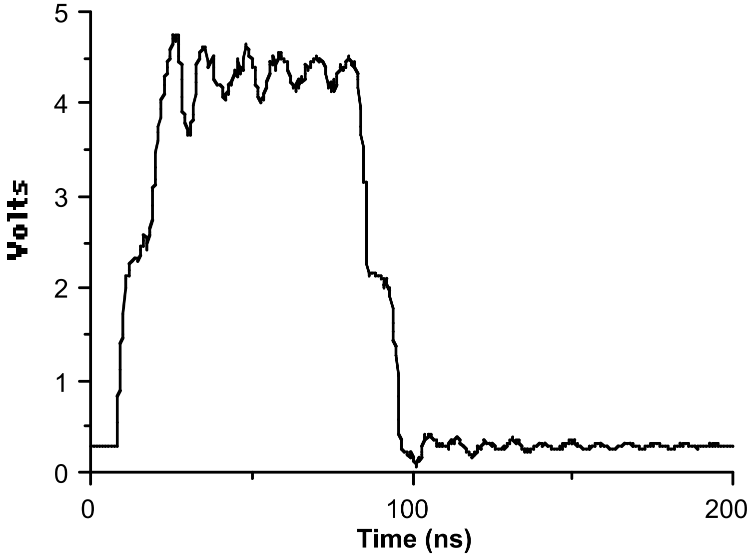


$$L_s = 312 \text{ nH/m}; \quad C_s = 100 \text{ pF/m};$$

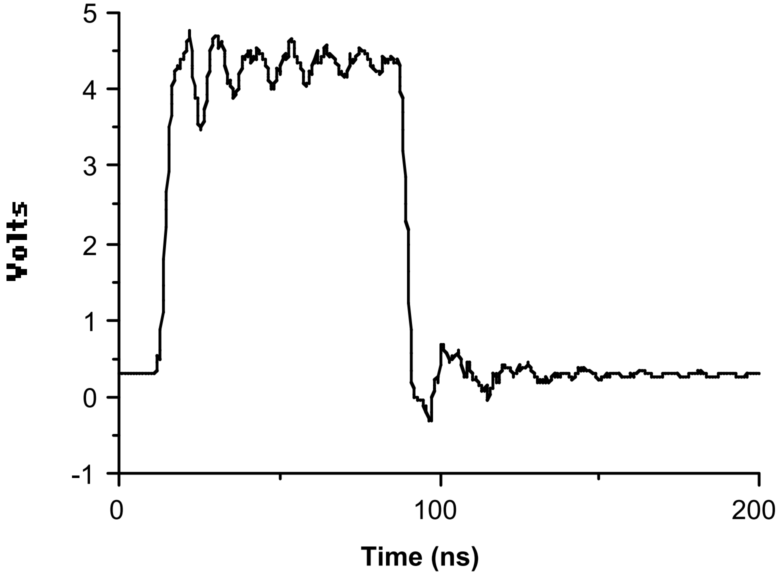
$$L_m = 85 \text{ nH/m}; \quad C_m = 12 \text{ pF/m}.$$

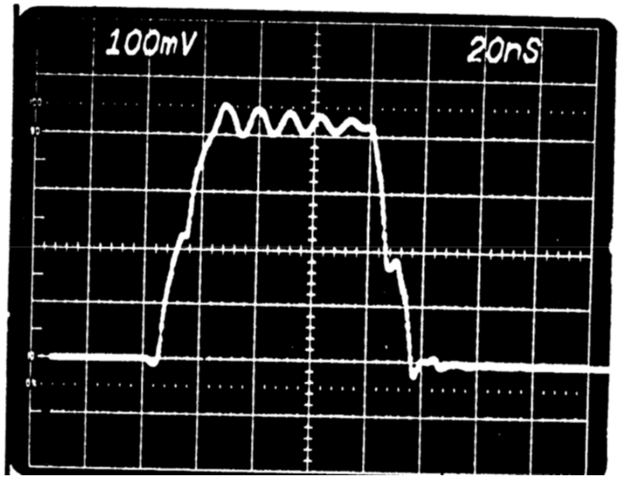
Drive Line 3

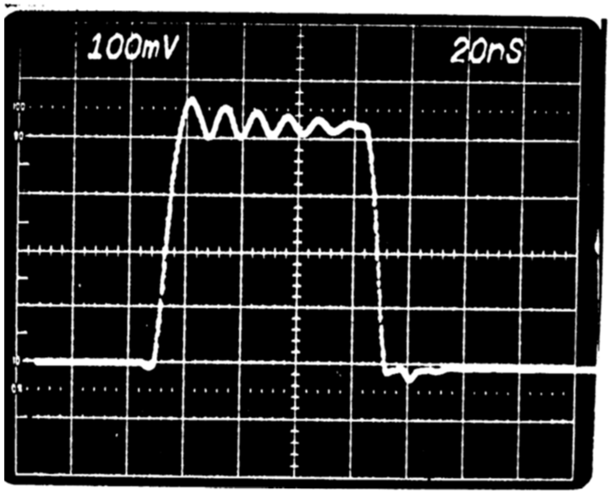
Drive line 3 at Near End



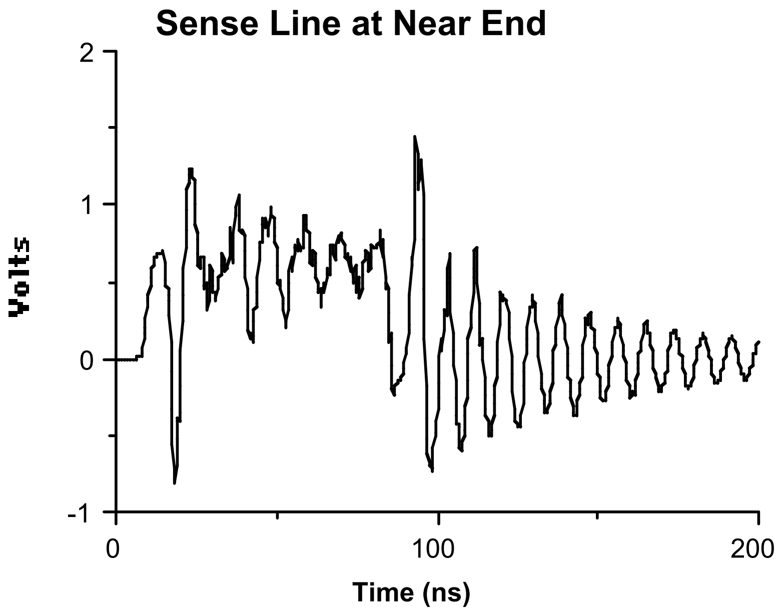
Drive Line 3 at Far End

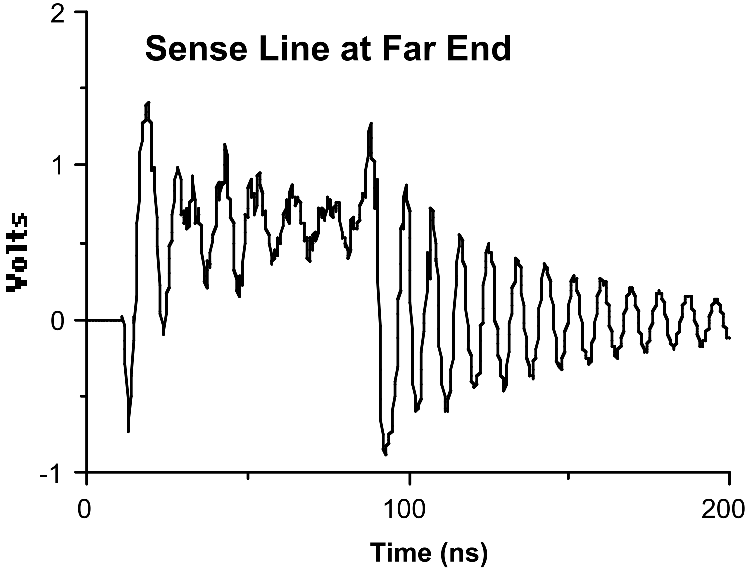


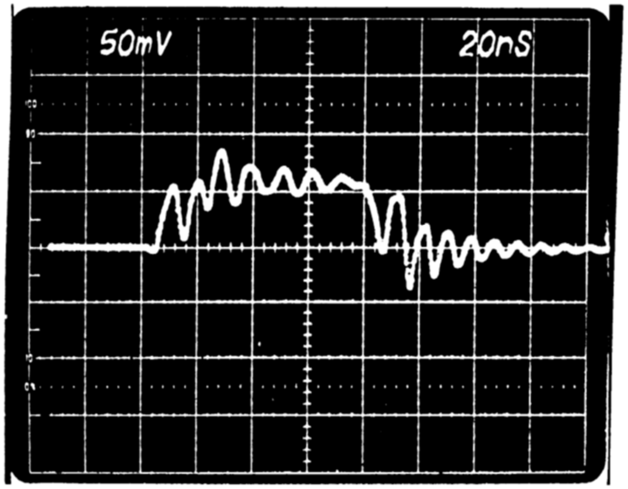


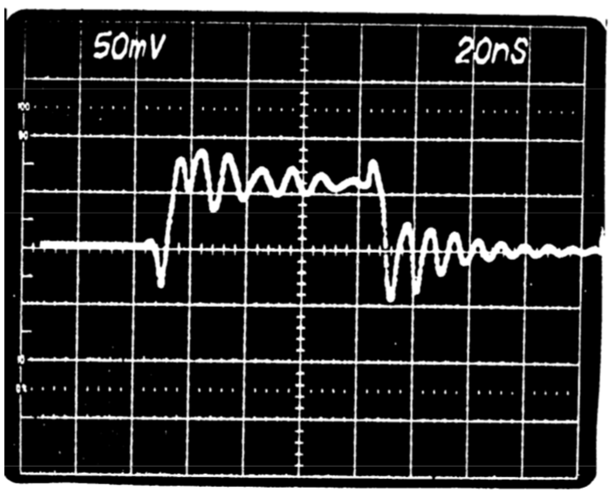


Sense Line

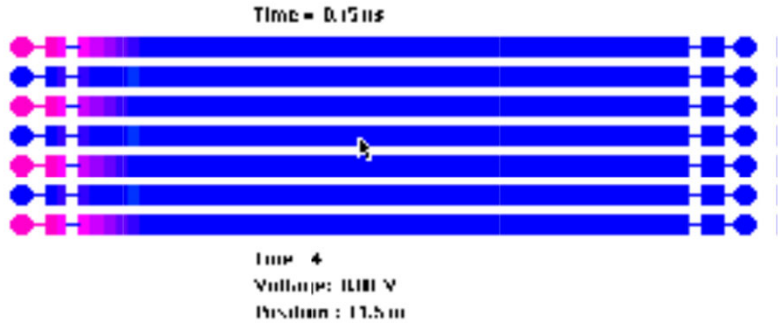

Sense Line at Near End


Sense Line at Far End

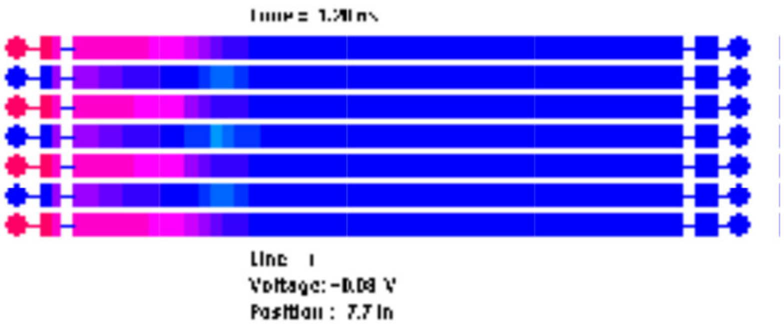




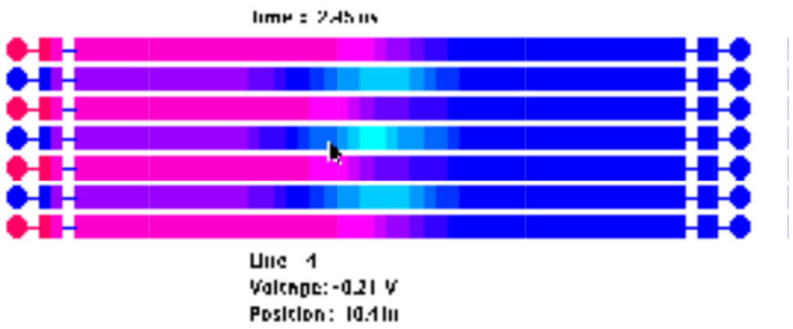
Multiconductor Simulation


Time = 0.15 ns

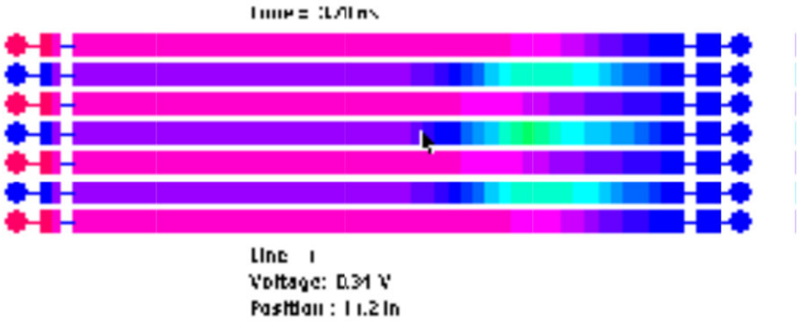
Line 4
Voltage: 0.00 V
Position : 11.5 in


Time = 1.20 ns

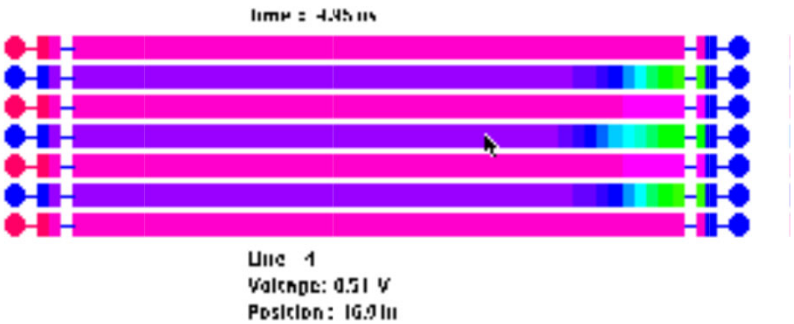
Line 1
Voltage: -0.08 V
Position : 7.7 in


Time = 2.45 ns

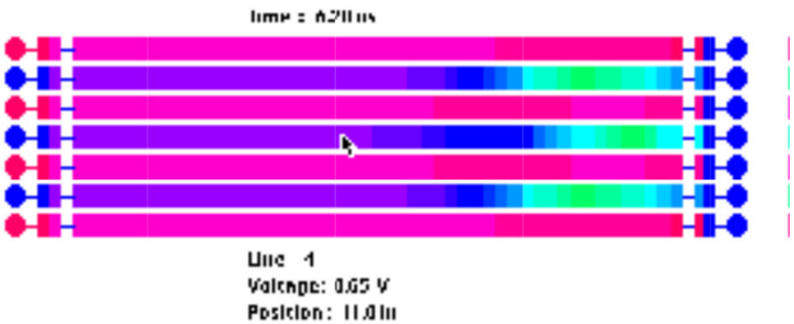
Line 4
Voltage: -0.21 V
Position : 10.4 in


Time = 3.70 ns

Line 1
Voltage: 0.34 V
Position : 11.2 in

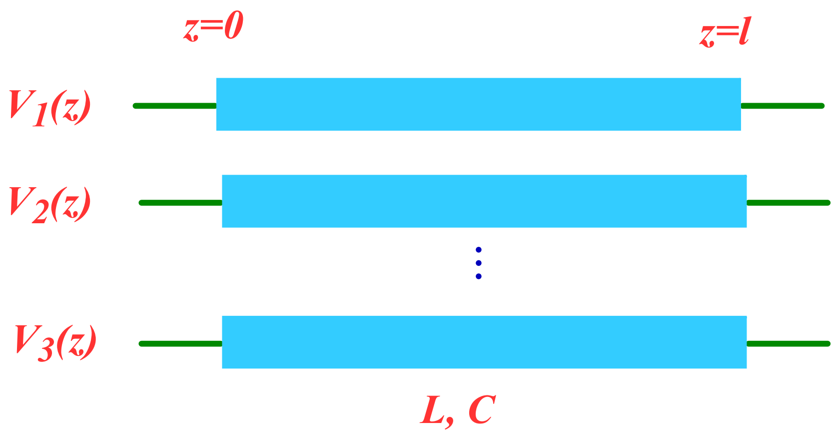

Time = 4.95 ns

Line 4
Voltage: 0.51 V
Position : 16.9 in


Time = 6.20 ns

Line 4
Voltage: 0.65 V
Position : 11.0 in

LOSSY COUPLED TRANSMISSION LINES

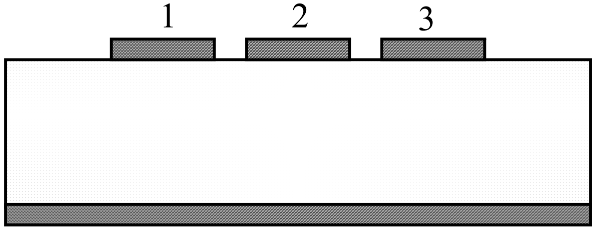


$$-\frac{\partial V}{\partial z} = RI + L\frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial z} = GV + C\frac{\partial V}{\partial t}$$

Solution is best found using a numerical approach (See References)

Three-Line Microstrip



$$-\frac{\partial V_1}{\partial z}=L_{11}\frac{\partial I_1}{\partial t}+L_{12}\frac{\partial I_2}{\partial t}+L_{13}\frac{\partial I_3}{\partial t}$$

$$-\frac{\partial I_1}{\partial z}=C_{11}\frac{\partial V_1}{\partial t}+C_{12}\frac{\partial V_2}{\partial t}+C_{13}\frac{\partial V_3}{\partial t}$$

$$-\frac{\partial V_2}{\partial z}=L_{21}\frac{\partial I_1}{\partial t}+L_{22}\frac{\partial I_2}{\partial t}+L_{23}\frac{\partial I_3}{\partial t}$$

$$-\frac{\partial I_2}{\partial z}=C_{21}\frac{\partial V_1}{\partial t}+C_{22}\frac{\partial V_2}{\partial t}+C_{23}\frac{\partial V_3}{\partial t}$$

$$-\frac{\partial V_3}{\partial z}=L_{31}\frac{\partial I_1}{\partial t}+L_{32}\frac{\partial I_2}{\partial t}+L_{33}\frac{\partial I_3}{\partial t}$$

$$-\frac{\partial I_3}{\partial z}=C_{31}\frac{\partial V_1}{\partial t}+C_{32}\frac{\partial V_2}{\partial t}+C_{33}\frac{\partial V_3}{\partial t}$$

Three-Line – Alpha Mode

Subtract (1c) from (1a) and (2c) from (2a), we get

$$-\frac{\partial V_\alpha}{\partial z} = (L_{11} - L_{13})\frac{\partial I_\alpha}{\partial t}$$
$$-\frac{\partial I_\alpha}{\partial z} = (C_{11} - C_{13})\frac{\partial V_\alpha}{\partial t}$$

This defines the Alpha mode with:

$V_\alpha = V_1 - V_3$ and $I_\alpha = I_1 - I_3$

The wave impedance of that mode is:

$$Z_\alpha = \sqrt{\frac{L_{11} - L_{13}}{C_{11} - C_{13}}}$$

and the velocity is $u_\alpha = \frac{1}{\sqrt{(L_{11} - L_{13})(C_{11} - C_{13})}}$

Three-Line – Modal Decomposition

In order to determine the next mode, assume that

$$V_\beta = V_1 + \beta V_2 + V_3$$

$$I_\beta = I_1 + \beta I_2 + I_3$$

$$-\frac{\partial V_\beta}{\partial z} = \left(L_{11} + \beta L_{21} + L_{31}\right)\frac{\partial I_1}{\partial t} + \left(L_{12} + \beta L_{22} + L_{32}\right)\frac{\partial I_2}{\partial t} + \left(L_{13} + \beta L_{23} + L_{33}\right)\frac{\partial I_3}{\partial t}$$

$$-\frac{\partial I_\beta}{\partial z} = \left(C_{11} + \beta C_{21} + C_{31}\right)\frac{\partial V_1}{\partial t} + \left(C_{12} + \beta C_{22} + C_{32}\right)\frac{\partial V_2}{\partial t} + \left(C_{13} + \beta C_{23} + C_{33}\right)\frac{\partial V_3}{\partial t}$$

By reciprocity $L_{32} = L_{23}$, $L_{21} = L_{12,}$ $L_{13} = L_{31}$

By symmetry, $L_{12} = L_{23}$

Also by approximation, $L_{22} \approx L_{11}$, $L_{11} + L_{13} \approx L_{11}$

Three-Line – Modal Decomposition

$$-\frac{\partial V_\beta}{\partial z}=\left(L_{11}+\beta L_{12}+L_{13}\right)\left(\frac{\partial I_1}{\partial t}+\frac{\partial I_3}{\partial t}\right)+\left(2L_{12}+\beta L_{11}\right)\frac{\partial I_2}{\partial t}$$

In order to balance the right-hand side into I_β, we need to have

$$\left(2L_{12}+\beta L_{11}\right)I_2=\beta\left(L_{11}+\beta L_{12}+L_{13}\right)I_2\approx\beta\left(L_{11}+\beta L_{12}\right)I_2$$

$$2L_{12}=\beta^2 L_{12}$$

or $\beta=\pm\sqrt{2}$

Therefore the other two modes are defined as

The Beta mode with

Three-Line – Beta Mode

The Beta mode with

$$V_\beta = V_1 + \sqrt{2}V_2 + V_3$$

$$I_\beta = I_1 + \sqrt{2}I_2 + I_3$$

The characteristic impedance of the Beta mode is:

$$Z_\beta = \sqrt{\frac{L_{11} + \sqrt{2}L_{12} + L_{13}}{C_{11} + \sqrt{2}C_{12} + C_{13}}}$$

and propagation velocity of the Beta mode is

$$u_\beta = \frac{1}{\sqrt{\left(L_{11} + \sqrt{2}L_{12} + L_{13}\right)\left(C_{11} + \sqrt{2}C_{12} + C_{13}\right)}}$$

Three-Line – Delta Mode

The Delta mode is defined such that

$$V_\delta = V_1 - \sqrt{2}V_2 + V_3$$

$$I_\delta = I_1 - \sqrt{2}I_2 + I_3$$

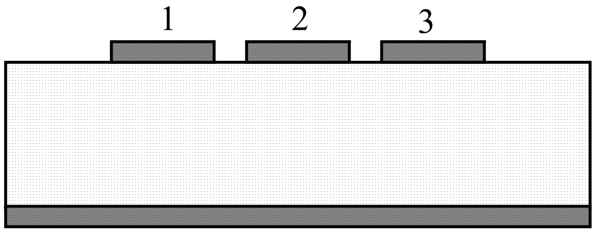
The characteristic impedance of the Delta mode is

$$Z_\delta = \sqrt{\frac{L_{11} - \sqrt{2}L_{12} + L_{13}}{C_{11} - \sqrt{2}C_{12} + C_{13}}}$$

The propagation velocity of the Delta mode is:

$$u_\delta = \frac{1}{\sqrt{\left(L_{11} - \sqrt{2}L_{12} + L_{13}\right)\left(C_{11} - \sqrt{2}C_{12} + C_{13}\right)}}$$

Symmetric 3-Line Microstrip



In summary: we have 3 modes for the 3-line system

$$E = \begin{pmatrix} 1 & 0 & -1 \\ 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

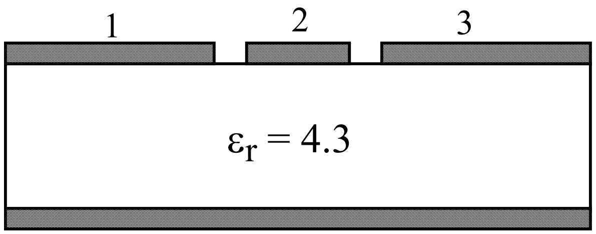
Alpha mode

Beta mode*

Delta mode*

*neglecting coupling between nonadjacent lines

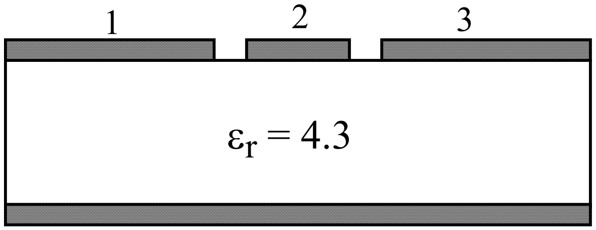
Coplanar Waveguide



$$L(nH/m) = \begin{pmatrix} 346 & 162 & 67 \\ 152 & 683 & 152 \\ 67 & 162 & 346 \end{pmatrix} \quad C(pF/m) = \begin{pmatrix} 113 & 17 & 5 \\ 16 & 53 & 16 \\ 5 & 17 & 113 \end{pmatrix}$$

$$E = \begin{pmatrix} 0.45 & 0.12 & 0.45 \\ 0.5 & 0 & -0.5 \\ -0.45 & 0.87 & -0.45 \end{pmatrix} \quad H = \begin{pmatrix} 0.44 & 0.49 & 0.44 \\ 0.5 & 0 & -0.5 \\ -0.10 & 0.88 & -0.10 \end{pmatrix}$$

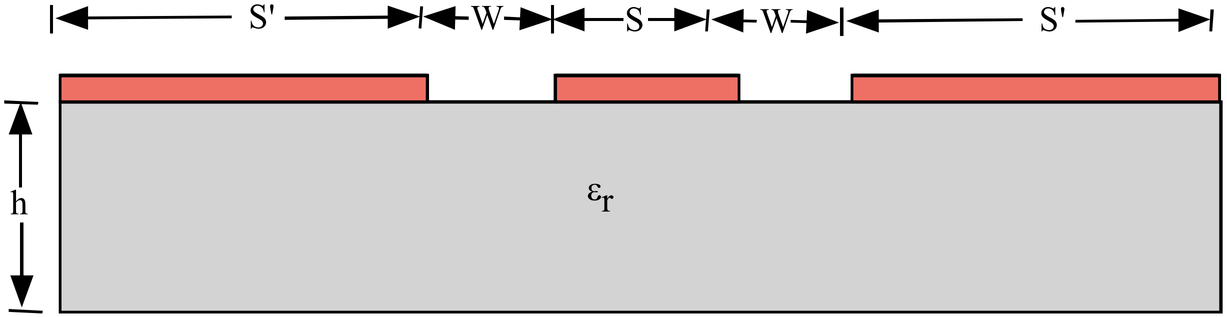
Coplanar Waveguide



$$Z_m(\Omega) = \begin{pmatrix} 73 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 94 \end{pmatrix} \qquad Z_c(\Omega) = \begin{pmatrix} 56 & 23 & 8 \\ 22 & 119 & 22 \\ 8 & 23 & 56 \end{pmatrix}$$

$$v_p(m/ns) = \begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.17 & 0 \\ 0 & 0 & 0.18 \end{pmatrix}$$

Coplanar Waveguide



$$K(k): \text{Complete Elliptic Integral of the first kind}$$

$$k = \frac{S}{S+2W}$$

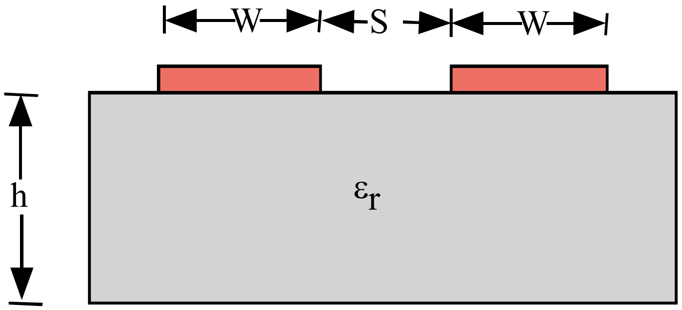
$$Z_{ocp} = \frac{30\pi}{\sqrt{\frac{\varepsilon_r+1}{2}}}\frac{K'(k)}{K(k)} \text{ (ohm)}$$

$$K'(k) = K(k')$$

$$k' = (1-k^2)^{1/2}$$

$$v_{cp} = \left(\frac{2}{\varepsilon_r+1}\right)^{1/2} c$$

Coplanar Strips



$$Z_{ocs} = \frac{120\pi}{\sqrt{\frac{\varepsilon_r + 1}{2}}} \frac{K'(k)}{K(k)} \text{ (ohm)}$$

Qualitative Comparison

Characteristic	**Microstrip**	**Coplanar Wguide**	**Coplanar strips**
ε_{eff}*	~6.5	~5	~5
Power handling	High	Medium	Medium
Radiation loss	Low	Medium	Medium
Unloaded Q	High	Medium	Low or High
Dispersion	Small	Medium	Medium
Mounting (shunt)	Hard	Easy	Easy
Mounting (series)	Easy	Easy	Easy

* Assuming ε_r=10 and h=0.025 inch