ECE 546
Lecture -13
Scattering Parameters

Spring 2024

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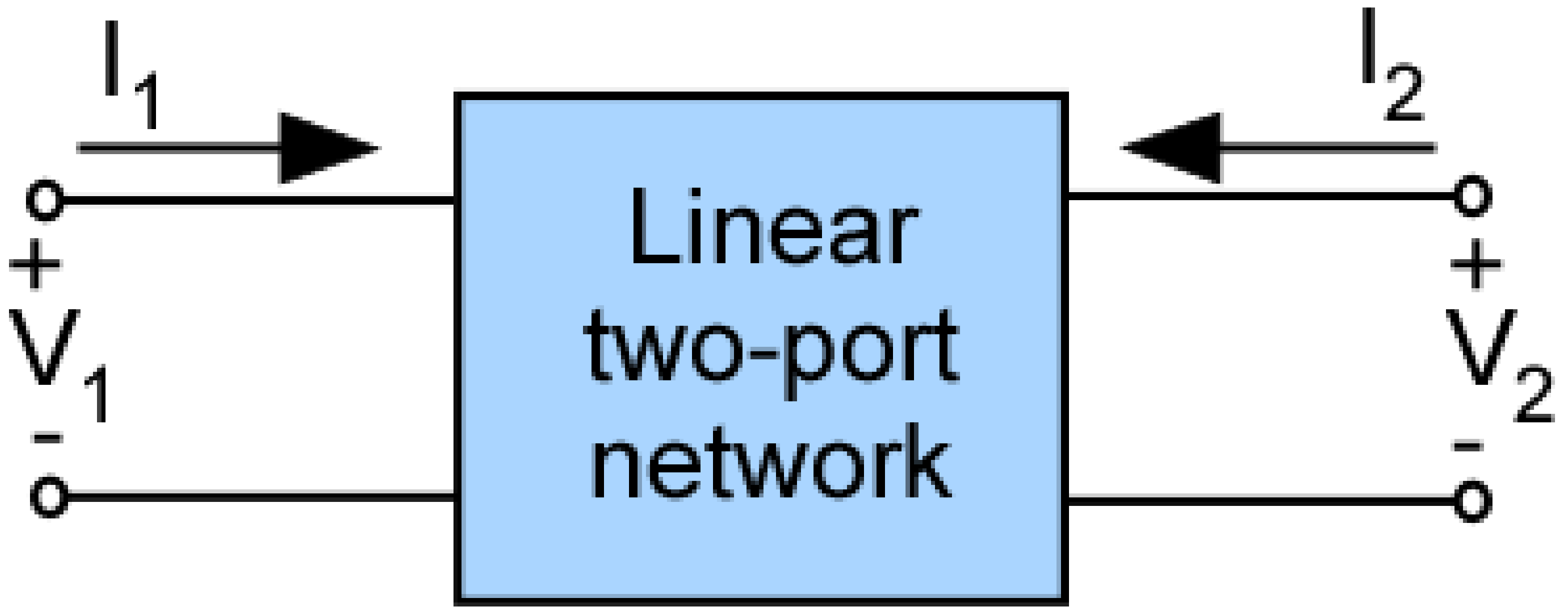
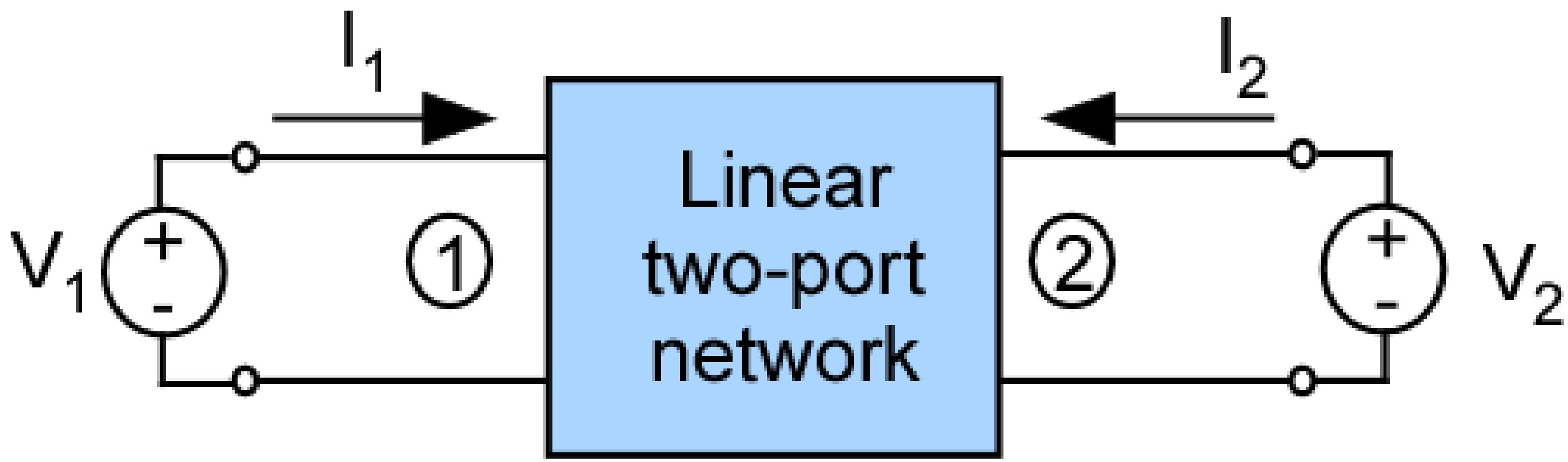
Transfer Function Representation



Use a two-terminal representation of system for input and output

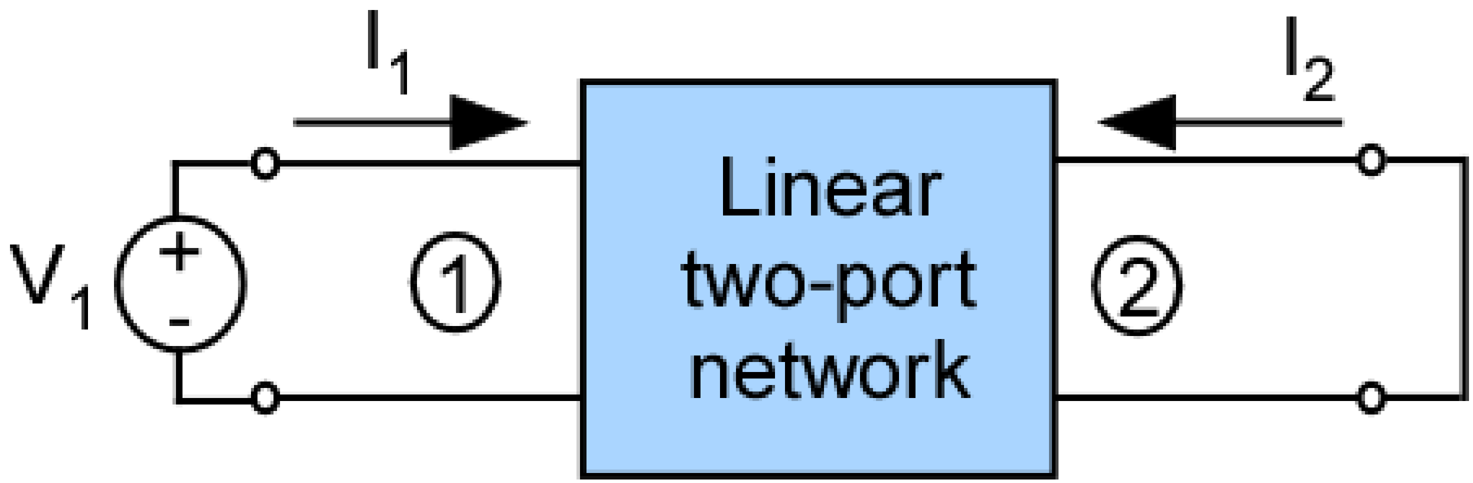
Y-parameter Representation



$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

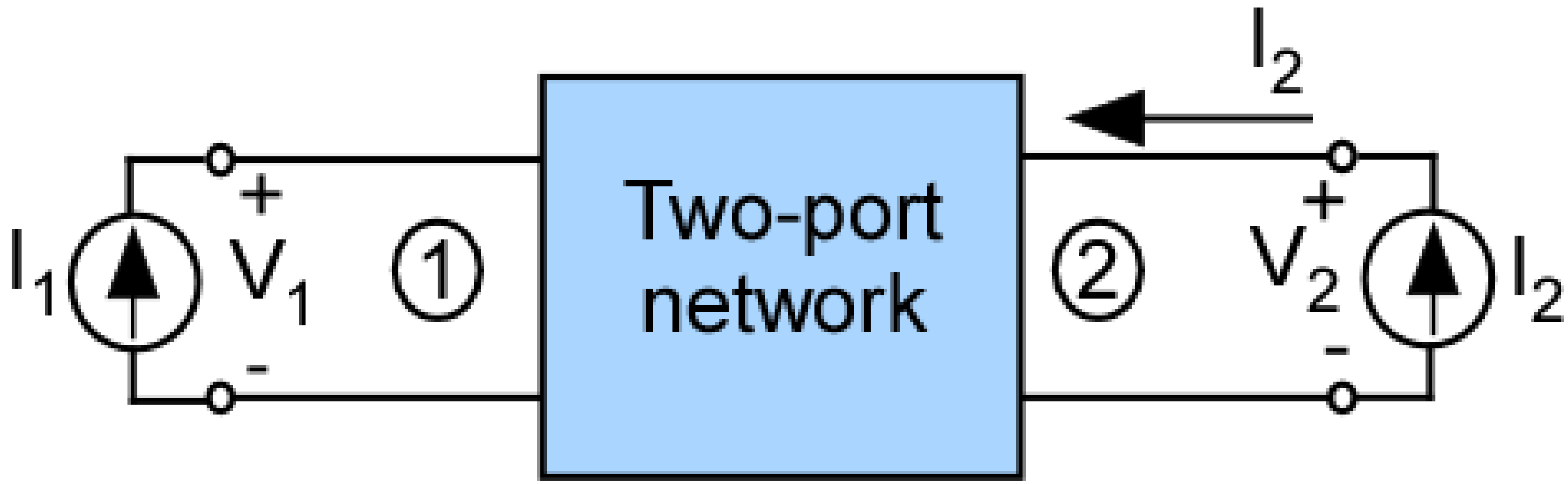
Y Parameter Calculations



$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \qquad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

To make $V_2 = 0$, place a short at port 2

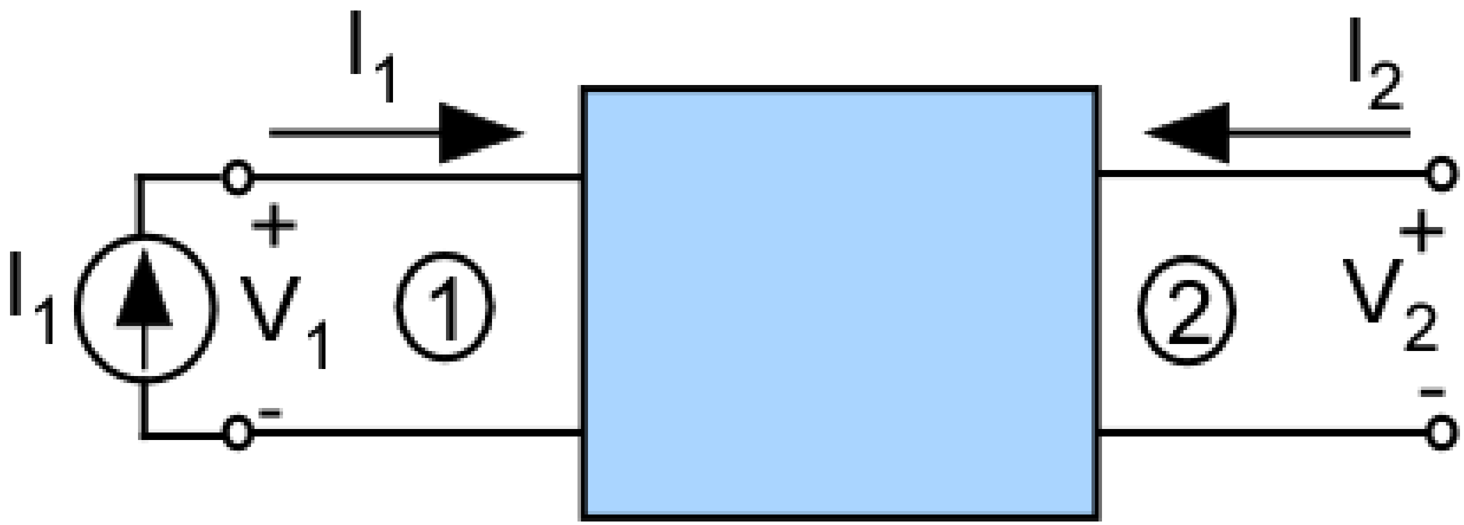
Z Parameters



$$V_1 = z_{11} I_1 + z_{12} I_2$$

$$V_2 = z_{21} I_1 + z_{22} I_2$$

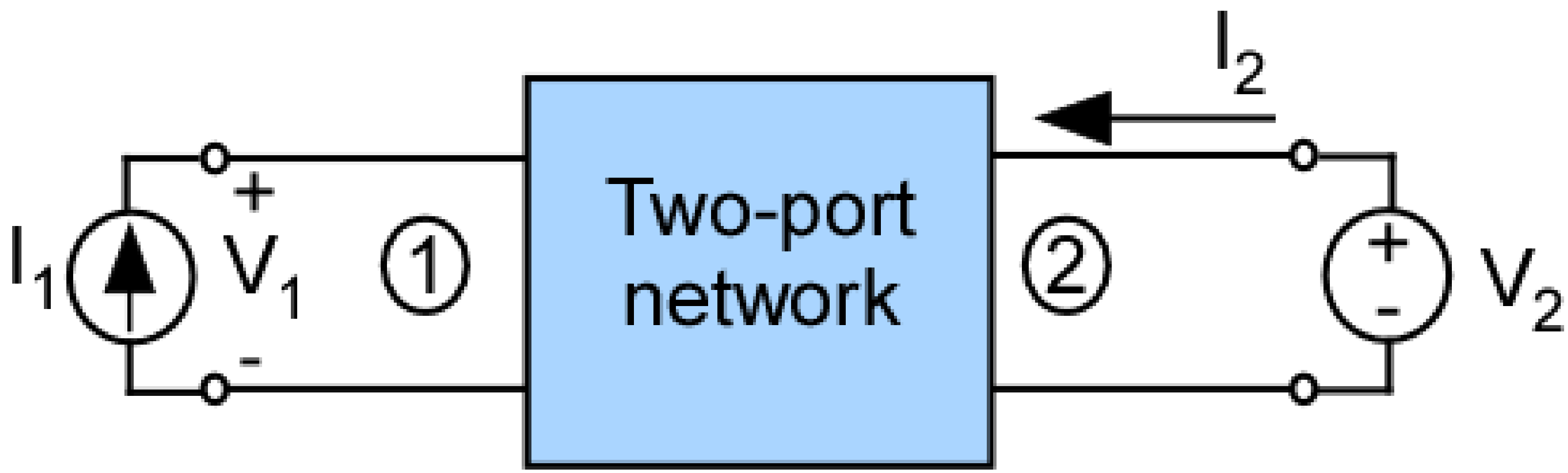
Z-parameter Calculations



$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

To make $I_2 = 0$, place an open at port 2

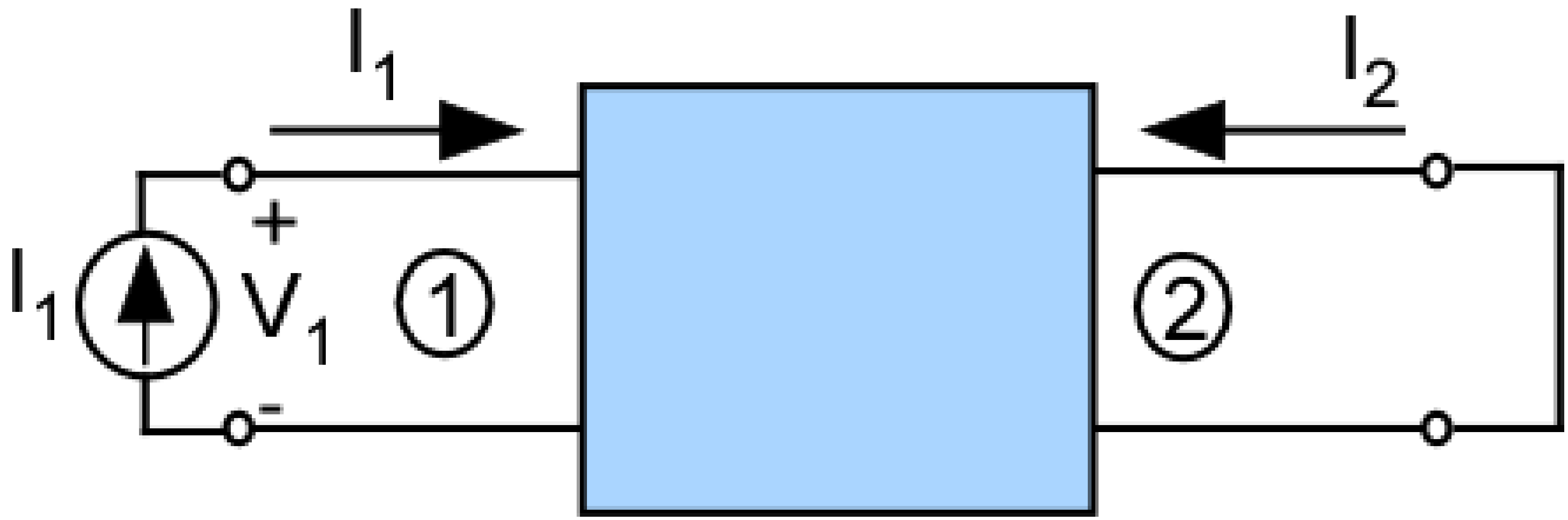
H Parameters



$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$I_2 = h_{21} I_1 + h_{22} V_2$$

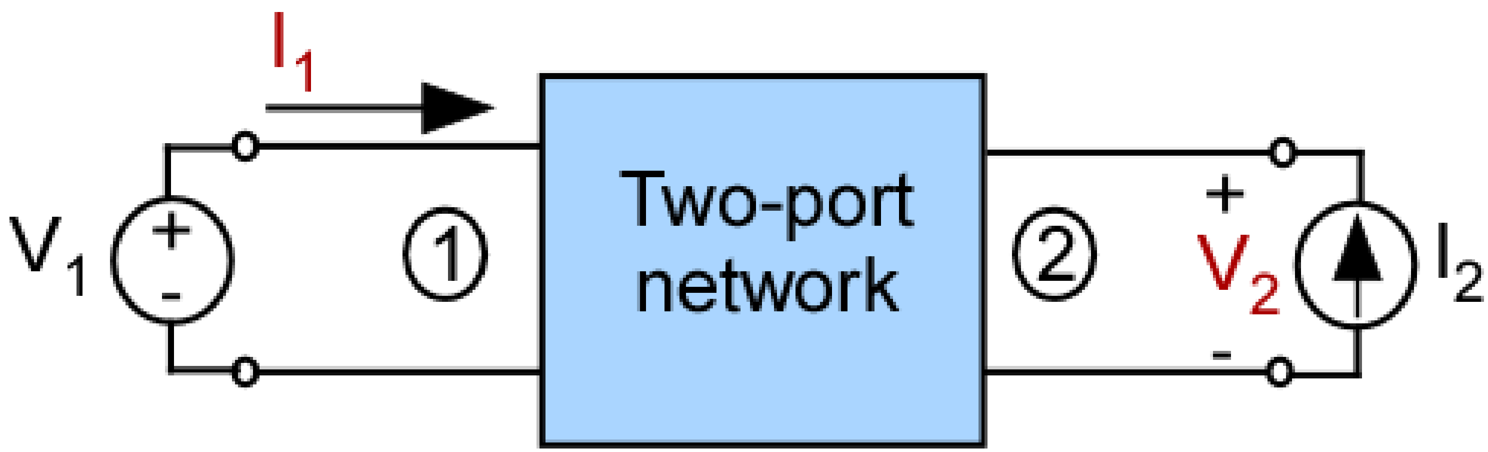
H Parameter Calculations



$$h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2=0} \qquad h_{21} = \left. \frac{I_2}{I_1} \right|_{V_2=0}$$

To make $V_2 = 0$, place a short at port 2

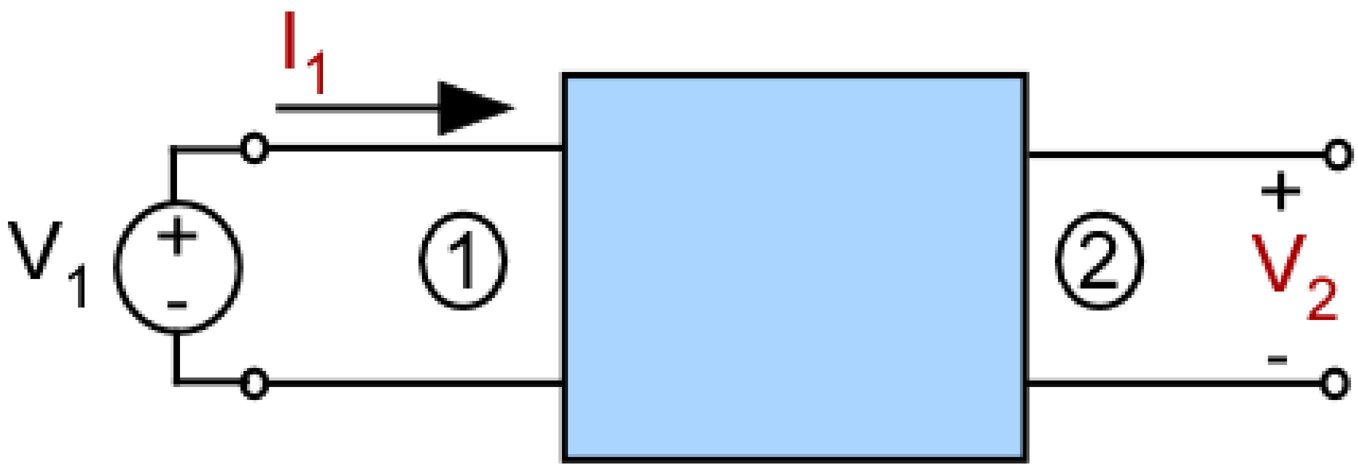
G Parameters



$$I_1 = g_{11}V_1 + g_{12}I_2$$

$$V_2 = g_{21}V_1 + g_{22}I_2$$

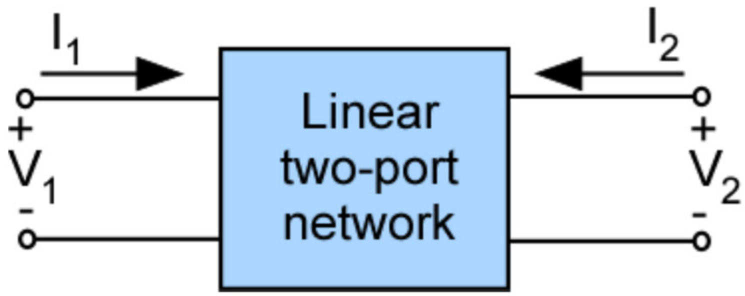
G-Parameter Calculations



$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} \qquad g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

To make $I_2 = 0$, place an open at port 2

TWO-PORT NETWORK REPRESENTATION



<u>Z Parameters</u>

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

<u>Y Parameters</u>

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

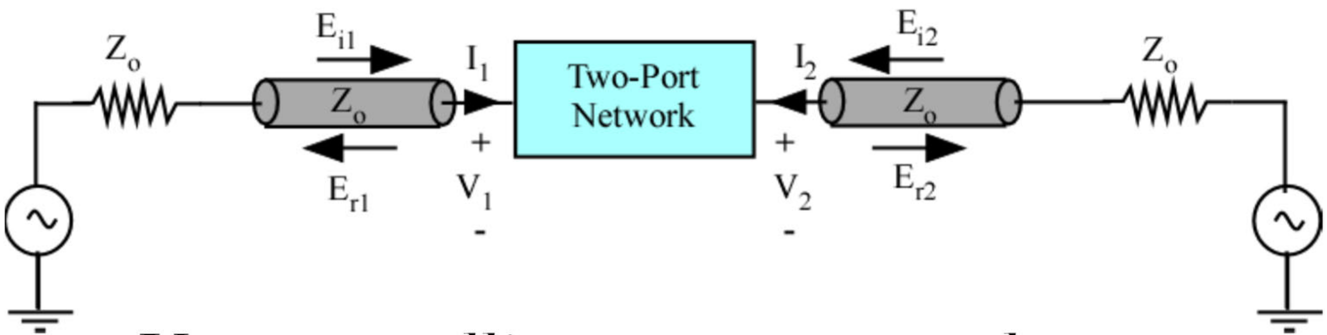
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

- At microwave frequencies, it is more difficult to measure total voltages and currents.

- Short and open circuits are difficult to achieve at high frequencies.

- Most active devices are not short- or open-circuit stable.

Wave Approach



Use a travelling wave approach

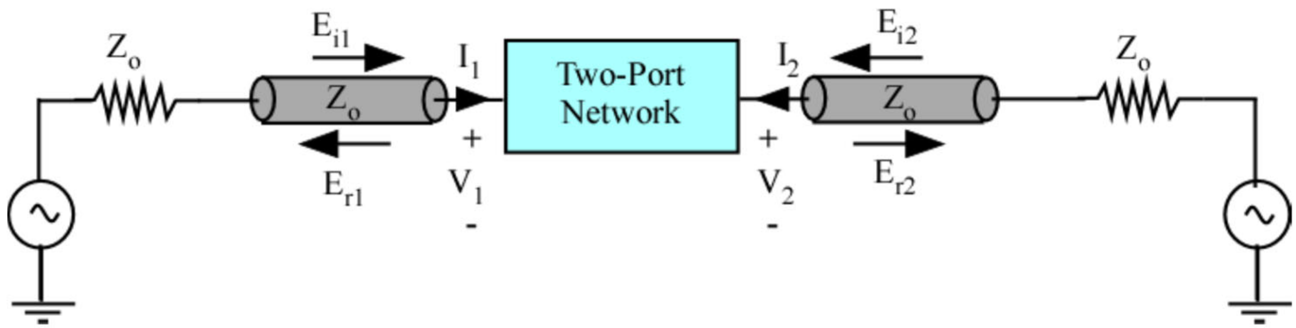
$$V_1 = E_{i1} + E_{r1} \qquad V_2 = E_{i2} + E_{r2}$$

$$\mathrm{I}_1 = \frac{E_{i1} - E_{r1}}{Z_o} \qquad \mathrm{I}_2 = \frac{E_{i2} - E_{r2}}{Z_o}$$

- Total voltage and current are made up of sums of forward and backward traveling waves.

- Traveling waves can be determined from standing-wave ratio.

Wave Approach



$$a_1 = \frac{E_{i1}}{\sqrt{Z_o}} \qquad a_2 = \frac{E_{i2}}{\sqrt{Z_o}}$$

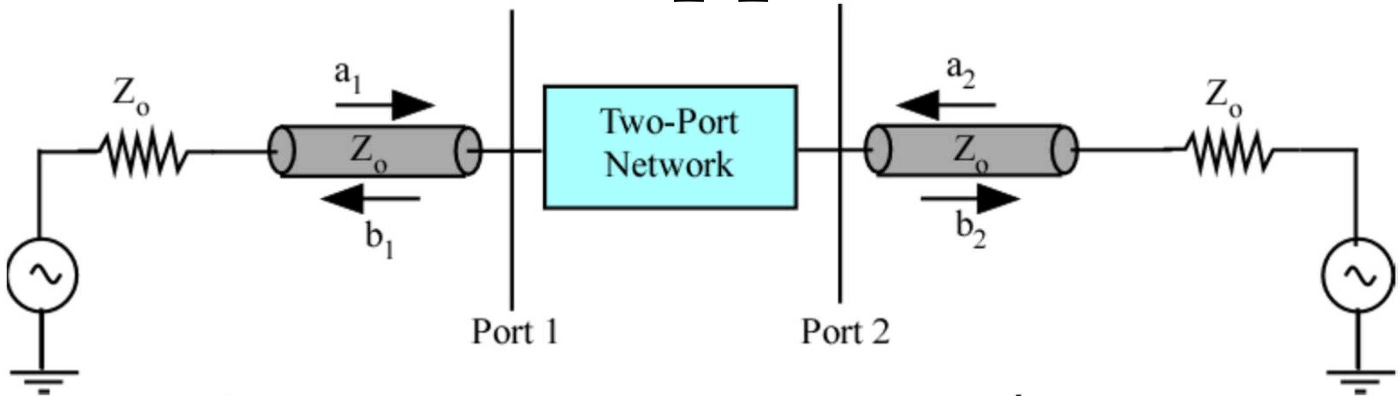
$$b_1 = \frac{E_{r1}}{\sqrt{Z_o}} \qquad b_2 = \frac{E_{r2}}{\sqrt{Z_o}}$$

Z_o is the reference impedance of the system

$$\mathbf{b_1 = S_{11}\, a_1 + S_{12}\, a_2}$$

$$\mathbf{b_2 = S_{21}\, a_1 + S_{22}\, a_2}$$

Wave Approach



$$S_{11} = \frac{b_1}{a_1}\bigg|_{a2=0} \qquad S_{21} = \frac{b_2}{a_1}\bigg|_{a2=0}$$

$$S_{12} = \frac{b_1}{a_2}\bigg|_{a1=0} \qquad S_{22} = \frac{b_2}{a_2}\bigg|_{a1=0}$$

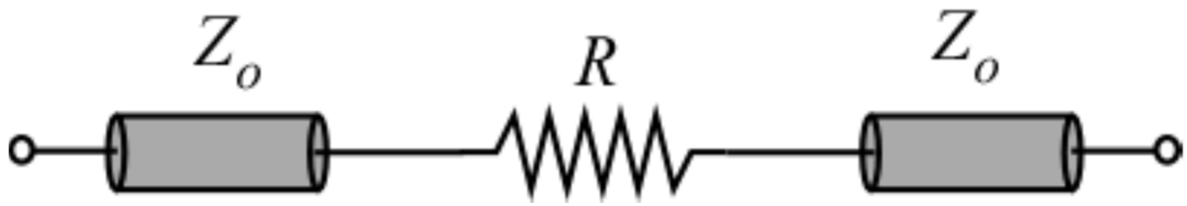
To make $a_i = 0$

1) Provide no excitation at port i

2) Match port i to the characteristic impedance of the reference lines.

CAUTION : a_i and b_i are the traveling waves in the <u>reference lines</u>.

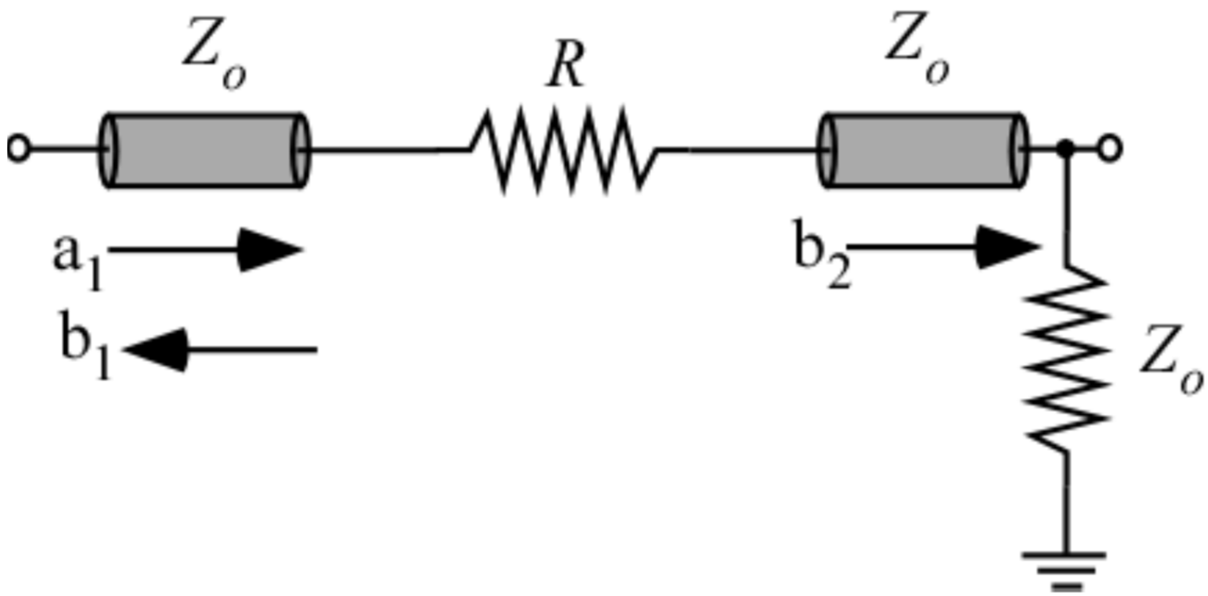
S-Parameters of Resistor



Determine S-Parameter of 2-port resistance

- Insert R between two reference TL
- Provide excitation at port 1 for S_{11} and S_{21}
- Provide excitation at port 2 for S_{12} and S_{23}
- Can use symmetry and reciprocity

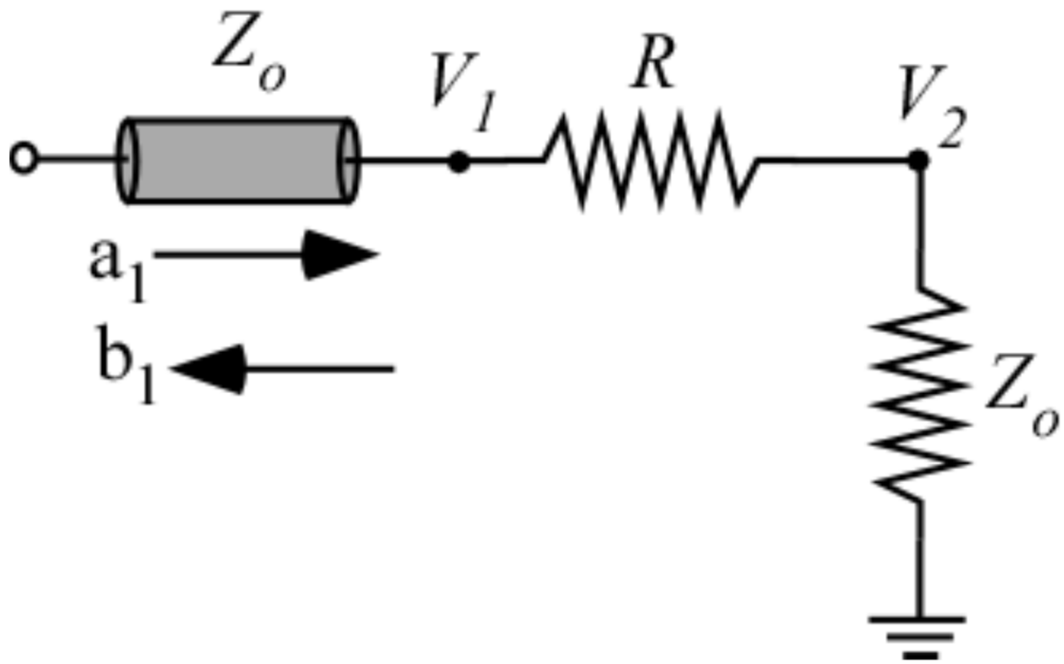
S-Parameters of Resistor



$$S_{11} = \frac{b_1}{a_1} = \Gamma = \frac{(R+Z_o)-Z_o}{(R+Z_o)+Z_o} = \frac{R}{R+2Z_o}$$

$$S_{11} = \frac{R}{R+2Z_o}$$ **and by symmetry,** $$S_{22} = \frac{R}{R+2Z_o}$$

Calculating S_{21} of Resistor

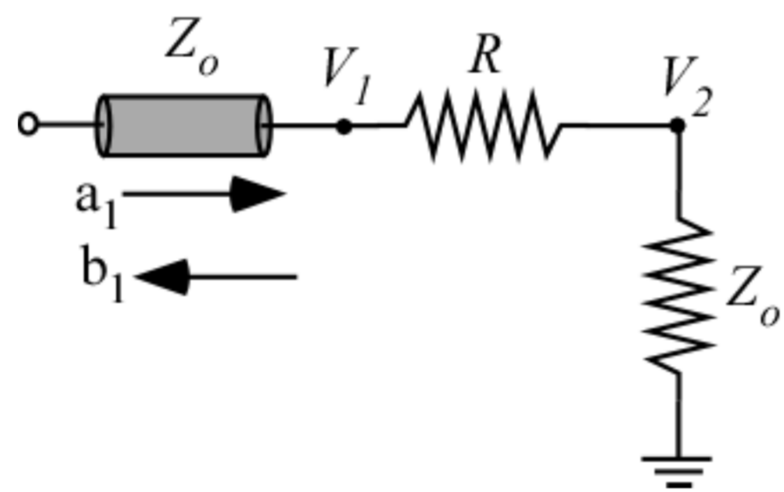


Since a_2=0, the total voltage in port 2 is: $V_2 = b_2\sqrt{Z_o}$

$$V_2 = \frac{V_1 Z_o}{R+Z_o} = \frac{\sqrt{Z_o}\left(a_1+b_1\right)Z_o}{R+Z_o} = \frac{\sqrt{Z_o}\left(a_1+S_{11}a_1\right)Z_o}{R+Z_o}$$

S-Parameters of Resistor

$$V_2 = \frac{Z_o\sqrt{Z_o}\left(1+S_{11}\right)a_1}{R+Z_o} = \frac{2Z_o a_1\sqrt{Z_o}}{R+2Z_o}$$



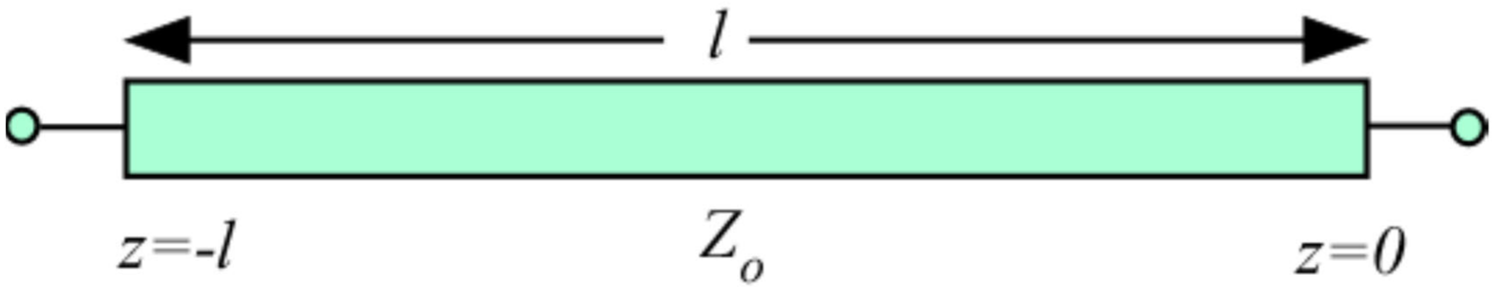
$$S_{21} = \frac{b_2}{a_1} = \frac{V_2}{\sqrt{Z_o}}\frac{1}{a_1} = \frac{2Z_o}{R+2Z_o}$$

$$S_{21} = \frac{2Z_o}{R+2Z_o}$$ **and by reciprocity,** $$S_{12} = \frac{2Z_o}{R+2Z_o}$$

S parameters of resistor R

$$S = \begin{bmatrix} \frac{R}{R+2Z_o} & \frac{2Z_o}{R+2Z_o} \\ \frac{2Z_o}{R+2Z_o} & \frac{R}{R+2Z_o} \end{bmatrix}$$

Y-Parameters of TL



at port 1

$$V_1 = V_+ e^{+j\beta l} + V_- e^{-j\beta l}$$

$$I_1 = Y_o\left(V_+ e^{+j\beta l} - V_- e^{-j\beta l}\right)$$

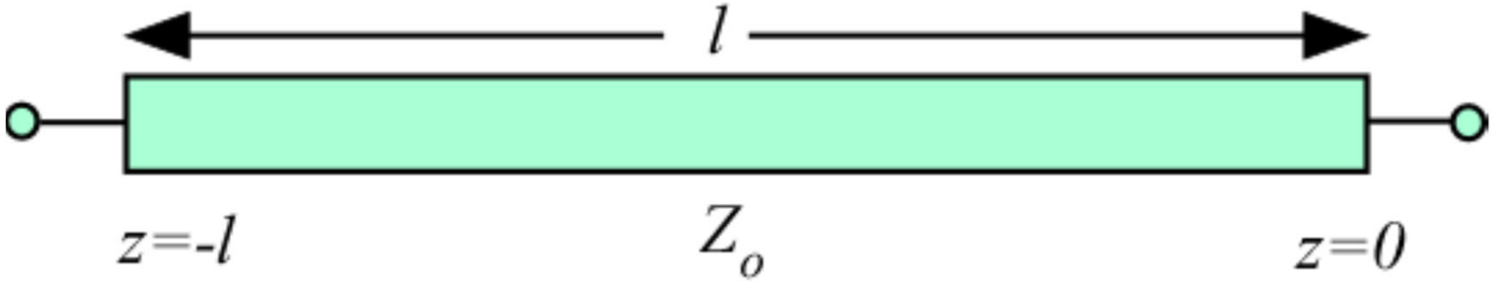
at port 2 ($z = 0$)

$$V_2 = V_+ + V_-$$

$$I_2 = -Y_o\left(V_+ - V_-\right)$$

$$V_+ = \frac{V_2 - Z_o I_2}{2} \quad \textbf{and} \quad V_- = \frac{V_2 + Z_o I_2}{2}$$

Y-Parameters of TL



So that

$$V_1 = \left(\frac{V_2 - Z_o I_2}{2}\right)e^{+j\beta l} + \left(\frac{V_2 + Z_o I_2}{2}\right)e^{-j\beta l}$$

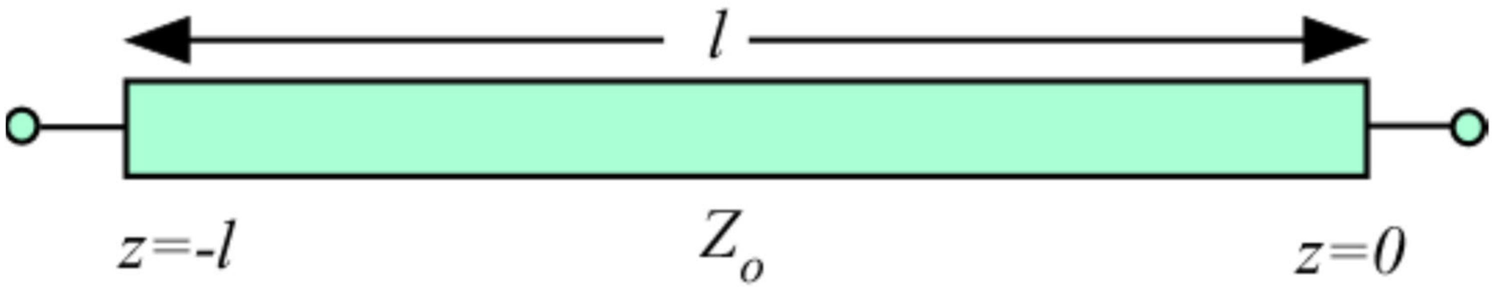
$$I_1 = Y_o\left(\frac{V_2 - Z_o I_2}{2}\right)e^{+j\beta l} - Y_o\left(\frac{V_2 + Z_o I_2}{2}\right)e^{-j\beta l}$$

and

$$V_1 = V_2 \cos\beta l - Z_o I_2 j \sin\beta l$$

$$I_1 = +Y_o V_2 j \sin\beta l - I_2 \cos\beta l$$

Y-Parameters of TL



Using definitions for Y_{11}

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{-I_2 \cos \beta l}{-jZ_o I_2 \sin \beta l} = \frac{-jY_o \cos \beta l}{\sin \beta l}$$

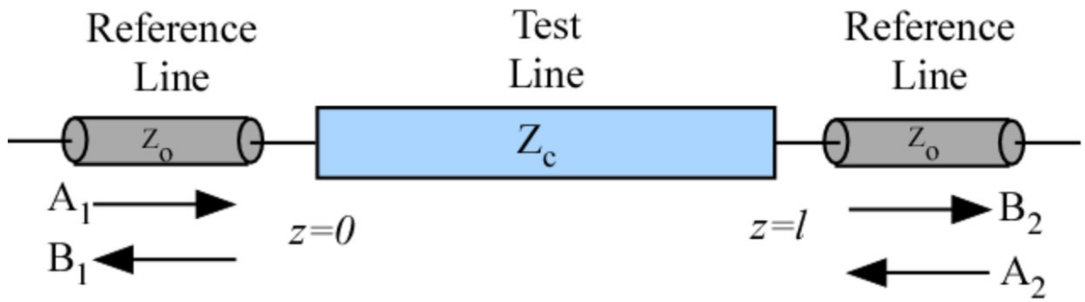
and

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-I_2}{-jZ_o I_2 \sin \beta l} = \frac{+jY_o}{\sin \beta l}$$

$$Y_{22} = Y_{11} \text{ by symmetry}$$

$$Y_{12} = Y_{21} \text{ by reciprocity}$$

S-Parameters of TL



$$S_{11} = S_{22} = \frac{(1-X^2)\Gamma}{1-X^2\Gamma^2}$$

$$S_{12} = S_{21} = \frac{(1-\Gamma^2)X}{1-X^2\Gamma^2}$$

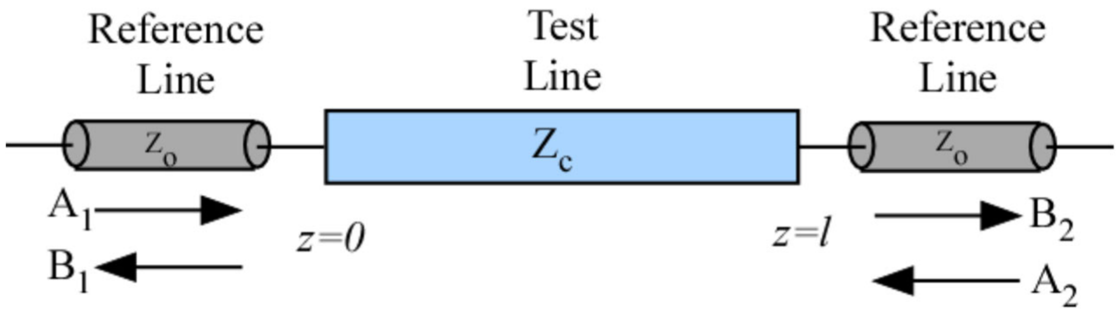
$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\Gamma = \frac{Z_c - Z_{ref}}{Z_c + Z_{ref}}$$

$$Z_c = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$X = e^{-\gamma l}$$

S-Parameters of Lossless TL



$$\beta = \omega\sqrt{LC}$$

$$S_{11} = S_{22} = \frac{(1-X^2)\Gamma}{1-X^2\Gamma^2}$$

$$\Gamma = \frac{Z_c - Z_{ref}}{Z_c + Z_{ref}}$$

$$Z_c = \sqrt{\frac{L}{C}}$$

$$S_{12} = S_{21} = \frac{(1-\Gamma^2)X}{1-X^2\Gamma^2}$$

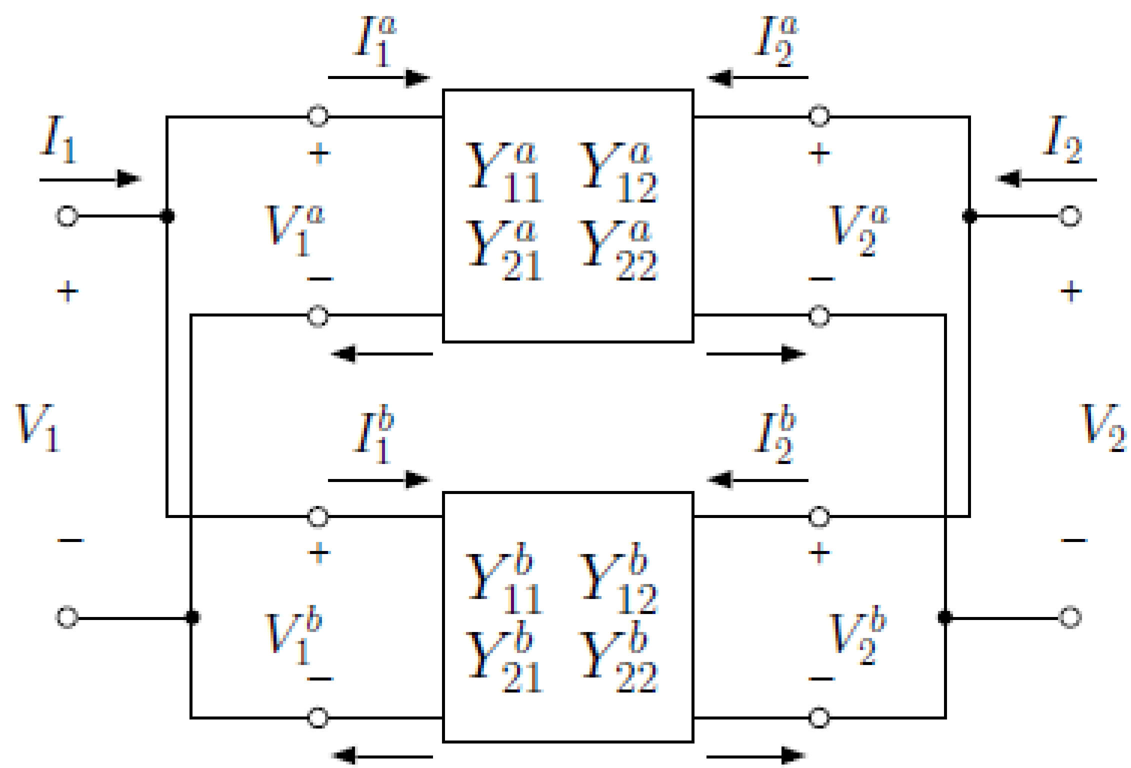
$$X = e^{-j\beta l}$$

If $Z_c = Z_{ref}$

$$S_{11} = S_{22} = 0$$

$$S_{12} = S_{21} = e^{-j\beta l}$$

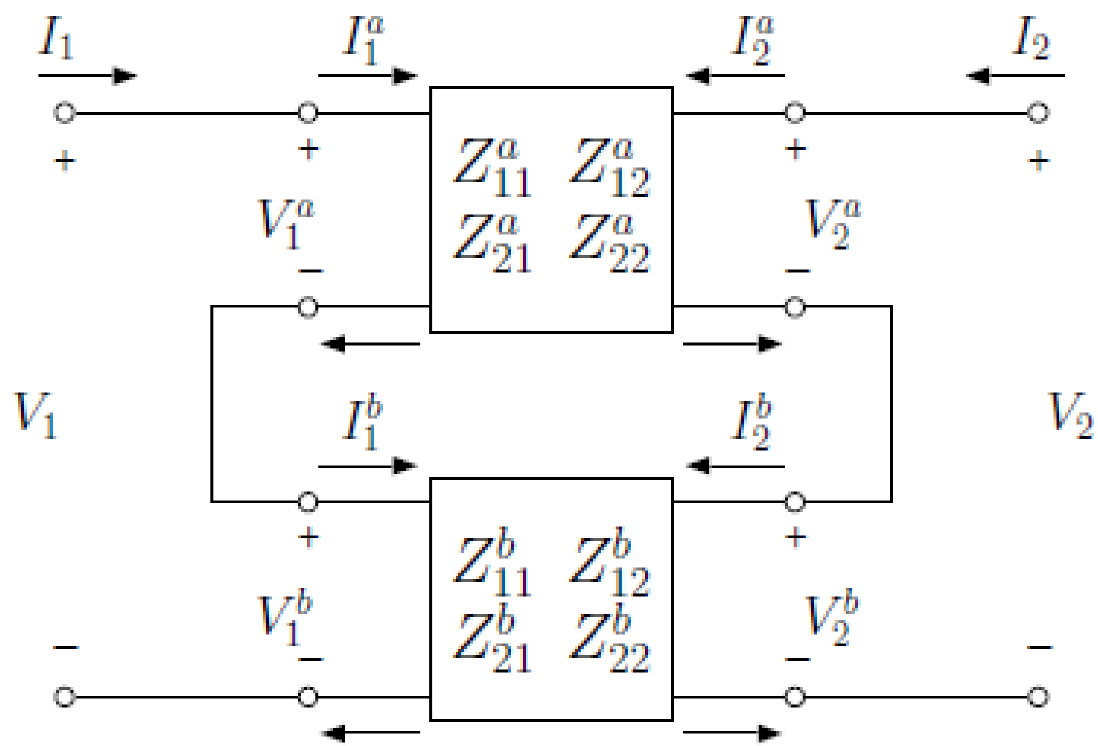
Two-Ports in Parallel*



$$\mathbf{Y} = \mathbf{Y}^{\mathbf{a}} + \mathbf{Y}^{\mathbf{b}}$$

* S. Franke, ECE 453 - *Wireless Communication Systems*, Spring 2019.

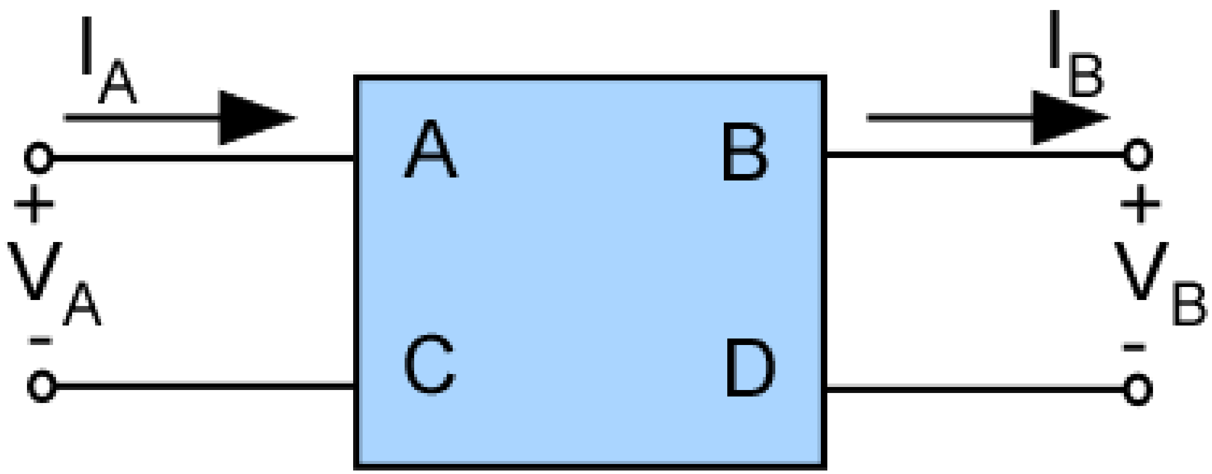
Two-Ports in Series*



$$\mathbf{Z} = \mathbf{Z}^a + \mathbf{Z}^b$$

* S. Franke, ECE 453 - *Wireless Communication Systems*, Spring 2019.

ABCD -Parameters



$$V_A = AV_B + BI_B$$

$$I_A = CV_B + DI_B$$

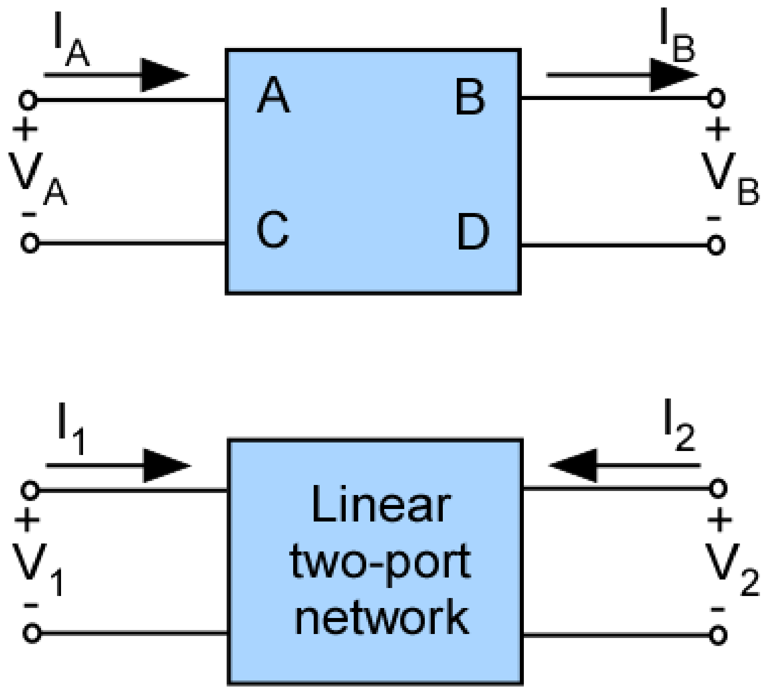
ABCD -Parameters

$$V_A = V_1$$

$$V_B = V_2$$

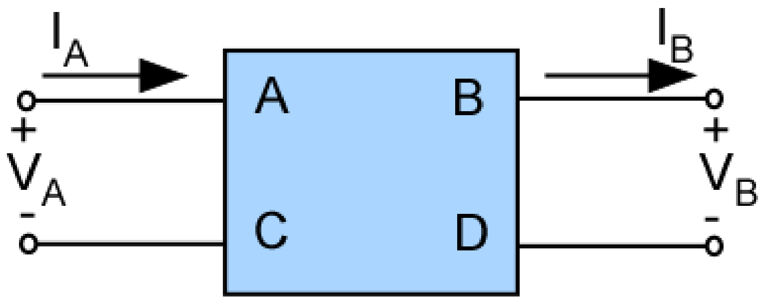
$$I_A = I_1$$

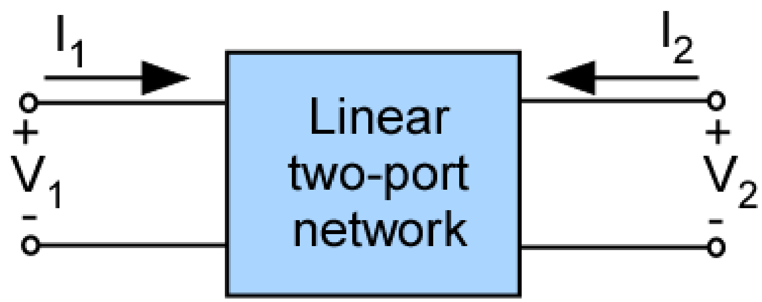
$$I_B = -I_2$$



Relationship with Z parameters is obtained by first expressing ABCD parameters in terms of Z parameters

ABCD -Parameters





From

$$V_A = Z_{11} I_A - Z_{12} I_B$$

$$V_B = Z_{21} I_A - Z_{22} I_B$$

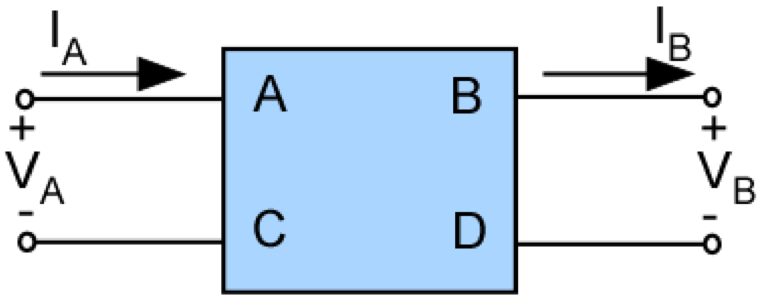
We get

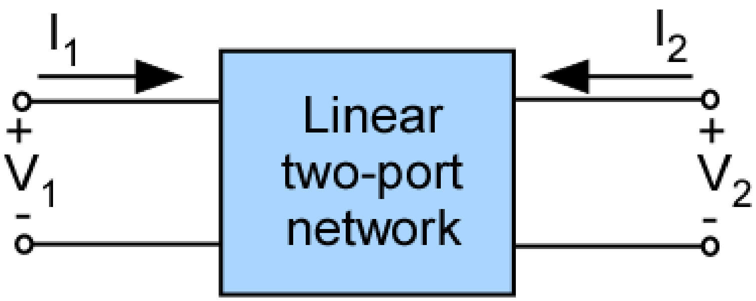
$$A = \frac{Z_{11}}{Z_{21}} \qquad B = \frac{\Delta}{Z_{21}}$$

$$C = \frac{1}{Z_{21}} \qquad D = \frac{Z_{22}}{Z_{21}}$$

$$\Delta = Z_{11} Z_{22} - Z_{12} Z_{21}$$

ABCD -Parameters





$$Z_{11} = \frac{A}{C} \qquad Z_{12} = \frac{(AD - BC)}{C}$$

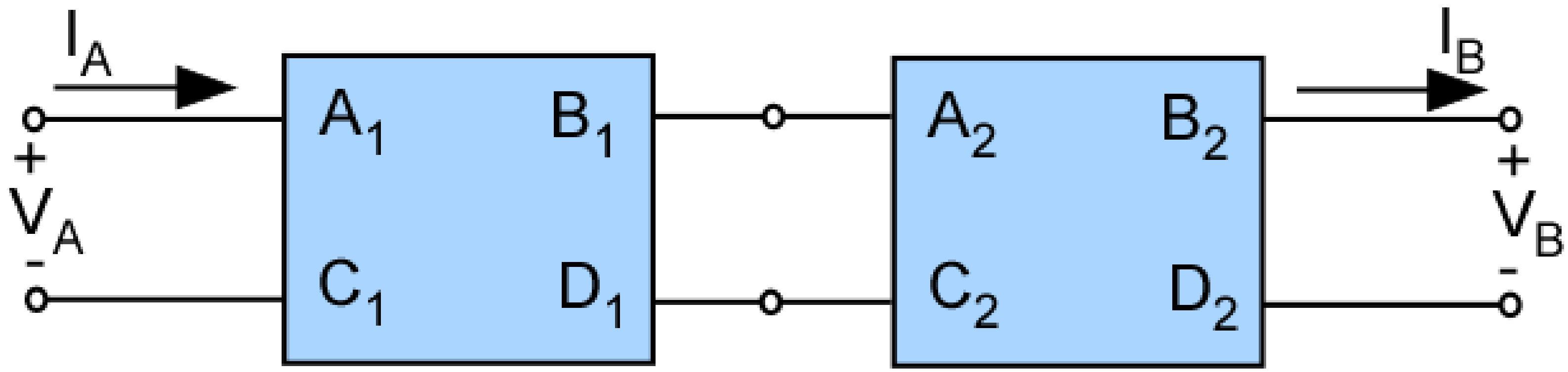
$$Z_{21} = \frac{1}{C} \qquad Z_{22} = \frac{D}{C}$$

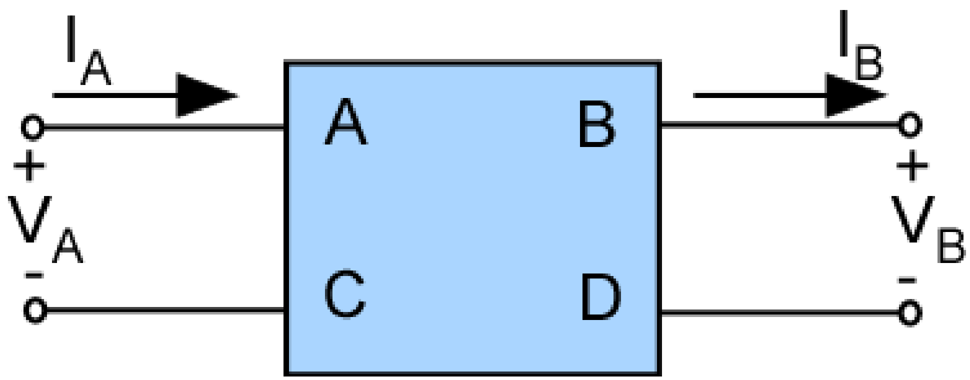
For a reciprocal network, $Z_{21} = Z_{12}$, therefore

$$AD - BC = 1$$

← **Reciprocity condition for ABCD parameters**

ABCD -Parameters





When cascading two-ports, it is best to use ABCD parameters. Put voltage and currents in cascadable form with the input variables in terms of the output variables

$$\mathbf{ABCD} = (\mathbf{ABCD})_1 \cdot (\mathbf{ABCD})_2$$

Scattering Transfer Parameters

In T-Parameters, traveling waves at the input are related to those at the output

$$b_1 = S_{11}a_1 + S_{12}a_2 \qquad b_1 = T_{11}a_2 + T_{12}b_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2 \qquad a_1 = T_{21}a_2 + T_{22}b_2$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} T_{12}T_{22}^{-1} & T_{11} - T_{12}T_{21}T_{22}^{-1} \\ T_{22}^{-1} & -T_{21}T_{22}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} S_{12} - S_{11}S_{22}S_{21}^{-1} & S_{11}S_{21}^{-1} \\ -S_{22}S_{21}^{-1} & S_{21}^{-1} \end{pmatrix}$$

T parameters can be cascaded $\mathbf{T} = \mathbf{T_A} \cdot \mathbf{T_B}$

Parameter Conversion*

7.7.1 Converting to Y-parameters

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{Z_{22}}{D_Z} & -\frac{Z_{12}}{D_Z} \\ -\frac{Z_{21}}{D_Z} & \frac{Z_{11}}{D_Z} \end{bmatrix} = \begin{bmatrix} \frac{1}{h_{11}} & -\frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{D_h}{h_{11}} \end{bmatrix} = \begin{bmatrix} \frac{D}{B} & -\frac{D_{ABCD}}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{bmatrix}$$

7.7.2 Converting to Z-parameters

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{Y_{22}}{D_Y} & -\frac{Y_{12}}{D_Y} \\ -\frac{Y_{21}}{D_Y} & \frac{Y_{11}}{D_Y} \end{bmatrix} = \begin{bmatrix} \frac{D_h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ -\frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix} = \begin{bmatrix} \frac{A}{C} & \frac{D_{ABCD}}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$$

7.7.3 Converting to h-parameters

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{D_Z}{Z_{22}} & \frac{Z_{12}}{Z_{22}} \\ -\frac{Z_{21}}{Z_{22}} & \frac{1}{Z_{22}} \end{bmatrix} = \begin{bmatrix} \frac{1}{Y_{11}} & -\frac{Y_{12}}{Y_{11}} \\ \frac{Y_{21}}{Y_{11}} & \frac{D_Y}{Y_{11}} \end{bmatrix} = \begin{bmatrix} \frac{B}{D} & \frac{D_{ABCD}}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$$

7.7.4 Converting to ABCD-parameters

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{Z_{11}}{Z_{21}} & \frac{D_Z}{Z_{21}} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{bmatrix} = \begin{bmatrix} -\frac{Y_{22}}{Y_{21}} & -\frac{1}{Y_{21}} \\ -\frac{D_Y}{Y_{21}} & -\frac{Y_{11}}{Y_{21}} \end{bmatrix} = \begin{bmatrix} -\frac{D_h}{h_{21}} & -\frac{h_{11}}{h_{21}} \\ -\frac{h_{22}}{h_{21}} & -\frac{1}{h_{21}} \end{bmatrix}$$

* M. Steer, *Microwave and RF Design*, Scitech Publishing, 2nd Edition 2013.

N-Port S Parameters

$$\begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ b_n \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdot & \cdot \\ S_{21} & S_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & S_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_n \end{bmatrix}$$

$$\mathbf{b} = \mathbf{S}\mathbf{a}$$

If $b_i = 0$, then no reflected wave on port *i* ➔ port is matched

$$a_i = \frac{V_i^+}{\sqrt{Z_{oi}}}$$

V_i^+ **: incident voltage wave in port *i***

V_i^- **: reflected voltage wave in port *i***

$$b_i = \frac{V_i^-}{\sqrt{Z_{oi}}}$$

Z_{oi} **: impedance in port *i***

N-Port S Parameters

$$\mathbf{v} = \sqrt{Z_o}(\mathbf{a}+\mathbf{b}) \quad (1) \qquad \mathbf{i} = \frac{1}{\sqrt{Z_o}}(\mathbf{a}-\mathbf{b}) \quad (2) \qquad \mathbf{v} = \mathbf{Zi} \quad (3)$$

Substitute (1) and (2) into (3)

$$\sqrt{Z_o}(\mathbf{a}+\mathbf{b}) = \mathbf{Z}\frac{1}{\sqrt{Z_o}}(\mathbf{a}-\mathbf{b})$$

Defining S such that b = Sa and substituting for b

$$Z_o(\mathbf{U}+\mathbf{S})\mathbf{a} = \mathbf{Z}(\mathbf{U}-\mathbf{S})\mathbf{a}$$

U : unit matrix

S→Z

$$\mathbf{Z} = Z_o(\mathbf{U}+\mathbf{S})(\mathbf{U}-\mathbf{S})^{-1}$$

Z→S

$$\mathbf{S} = (\mathbf{Z}+Z_o\mathbf{U})^{-1}(\mathbf{Z}-Z_o\mathbf{U})$$

N-Port S Parameters

If the port reference impedances are different, we define k as

$$\mathbf{k} = \begin{bmatrix} \sqrt{Z_{o1}} & & & \\ & \sqrt{Z_{o2}} & & \\ & & \cdot & \\ & & & \sqrt{Z_{on}} \end{bmatrix}$$

$\mathbf{v} = \mathbf{k}(\mathbf{a}+\mathbf{b})$ and $\mathbf{i} = \mathbf{k}^{-1}(\mathbf{a}-\mathbf{b})$ and $\mathbf{k}(\mathbf{a}+\mathbf{b}) = \mathbf{Z}\mathbf{k}^{-1}(\mathbf{a}-\mathbf{b})$

Z→S

$$\mathbf{S} = \left(\mathbf{Z}\mathbf{k}^{-1}+\mathbf{k}\right)\left(\mathbf{Z}\mathbf{k}^{-1}-\mathbf{k}\right)$$

S→Z

$$\mathbf{Z} = \mathbf{k}\left(\mathbf{U}+\mathbf{S}\right)\left(\mathbf{U}-\mathbf{S}\right)^{-1}\mathbf{k}$$

Normalization

Assume original S parameters as S_1 with system k_1. Then the representation S_2 on system k_2 is given by

Transformation Equation

$$S_2 = \left[k_1 (U + S_1)(U - S_1)^{-1} k_1 k_2 + k_2 \right]^{-1} \left[k_1 (U + S_1)(U - S_1)^{-1} k_1 k_2 - k_2 \right]$$

If Z is symmetric, S is also symmetric

Dissipated Power

$$P_d = \frac{1}{2}\mathbf{a}^{\mathrm{T}}\left(\mathbf{U} - \mathbf{S}^{\mathrm{T}}\mathbf{S}^*\right)\mathbf{a}^*$$

The dissipation matrix D is given by:

$$\mathbf{D} = \mathbf{U} - \mathbf{S}^{\mathrm{T}}\mathbf{S}^*$$

Passivity insures that the system will always be stable provided that it is connected to another passive network

For passivity
- **(1) the determinant of D must be** ≥ 0
- **(2) the determinant of the principal minors must be** ≥ 0

Dissipated Power

When the dissipation matrix is 0, we have a lossless network→

$$\mathbf{S}^T\mathbf{S}^* = \mathbf{U}$$

The S matrix is unitary.

For a lossless two-port:

$$|S_{11}|^2 + |S_{21}|^2 = 1$$

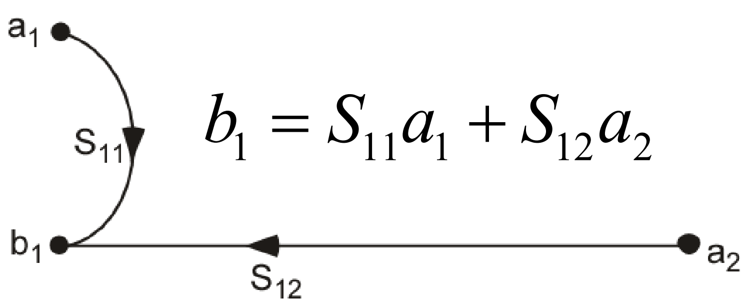
$$|S_{22}|^2 + |S_{12}|^2 = 1$$

If in addition the network is reciprocal, then

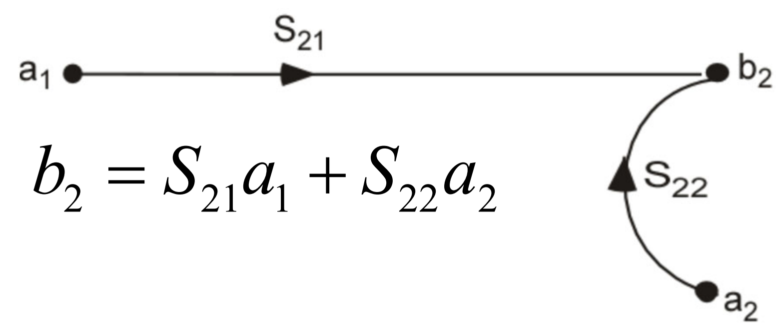
$$S_{12} = S_{21} \text{ and } |S_{11}| = |S_{22}| = \sqrt{1 - |S_{12}|^2}$$

Flow Graph Definitions

- Voltage waves designated as nodes.
- S parameters designated as branches
- Branches enter dependent nodes and emanate from independent nodes

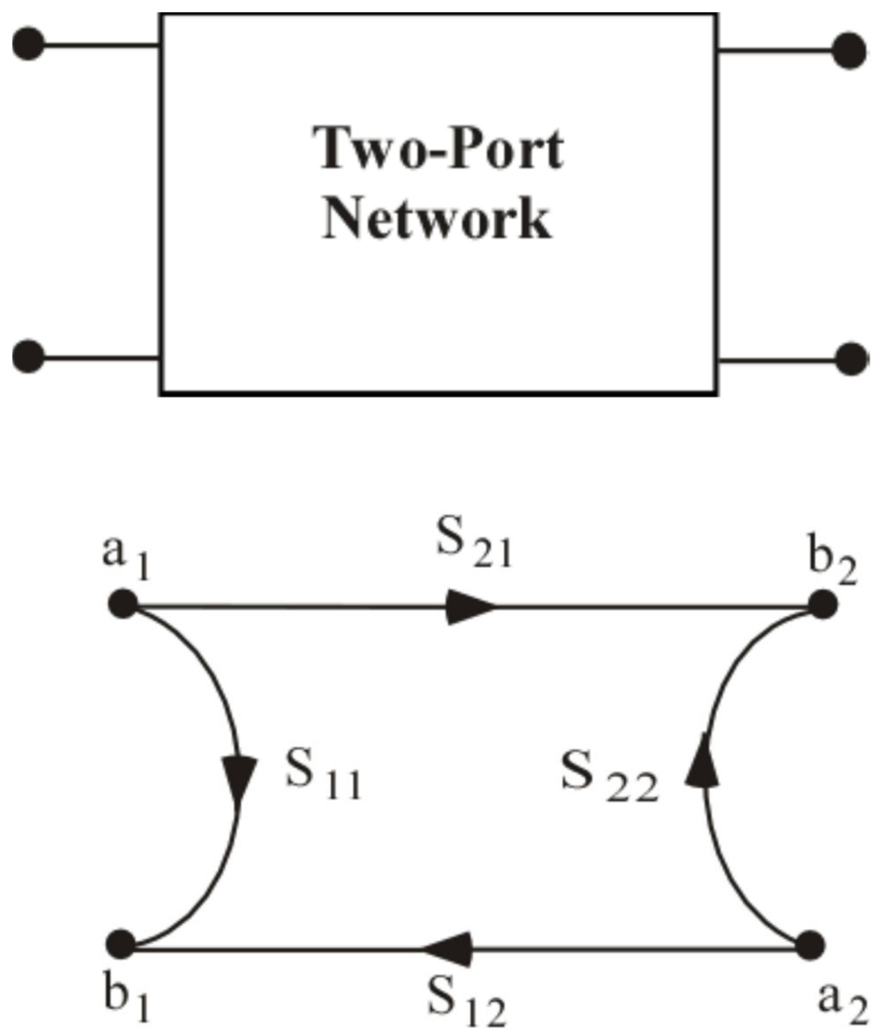


$$b_1 = S_{11}a_1 + S_{12}a_2$$

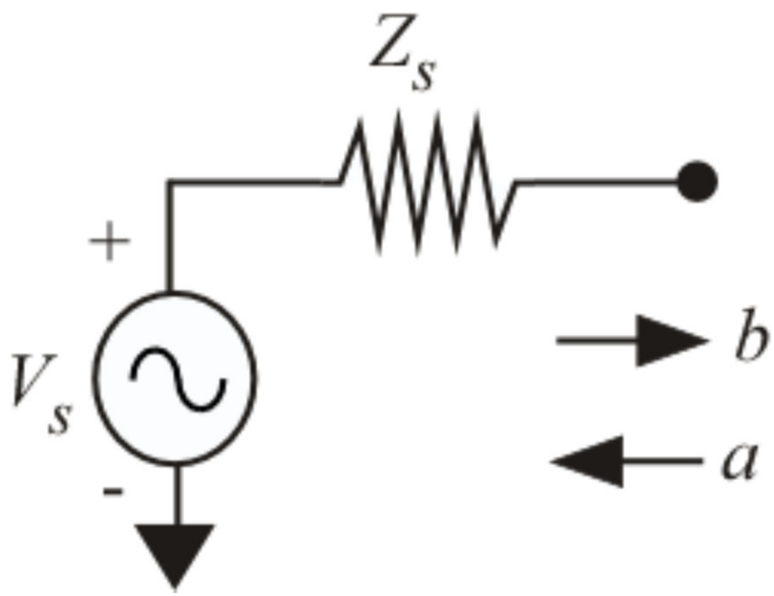


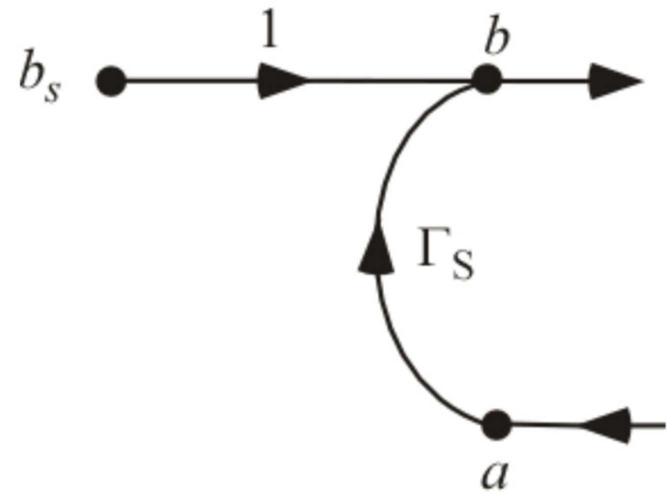
$$b_2 = S_{21}a_1 + S_{22}a_2$$

Two-Port Flow Graph



Flow Graph for Source



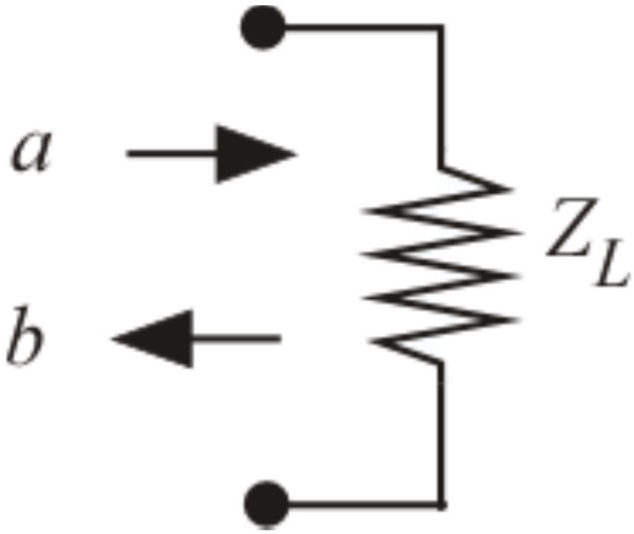


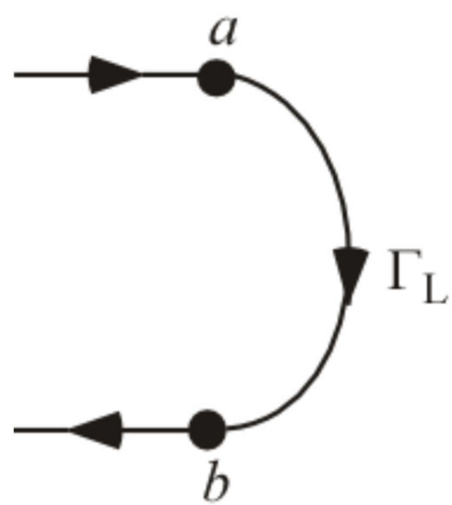
$$b_S = \frac{V_S\sqrt{Z_o}}{Z_S + Z_o}$$

$$\Gamma_S = \frac{Z_S - Z_o}{Z_S + Z_o}$$

b_s is power wave associated with power dissipated in a load of value Z_o connected to the source.

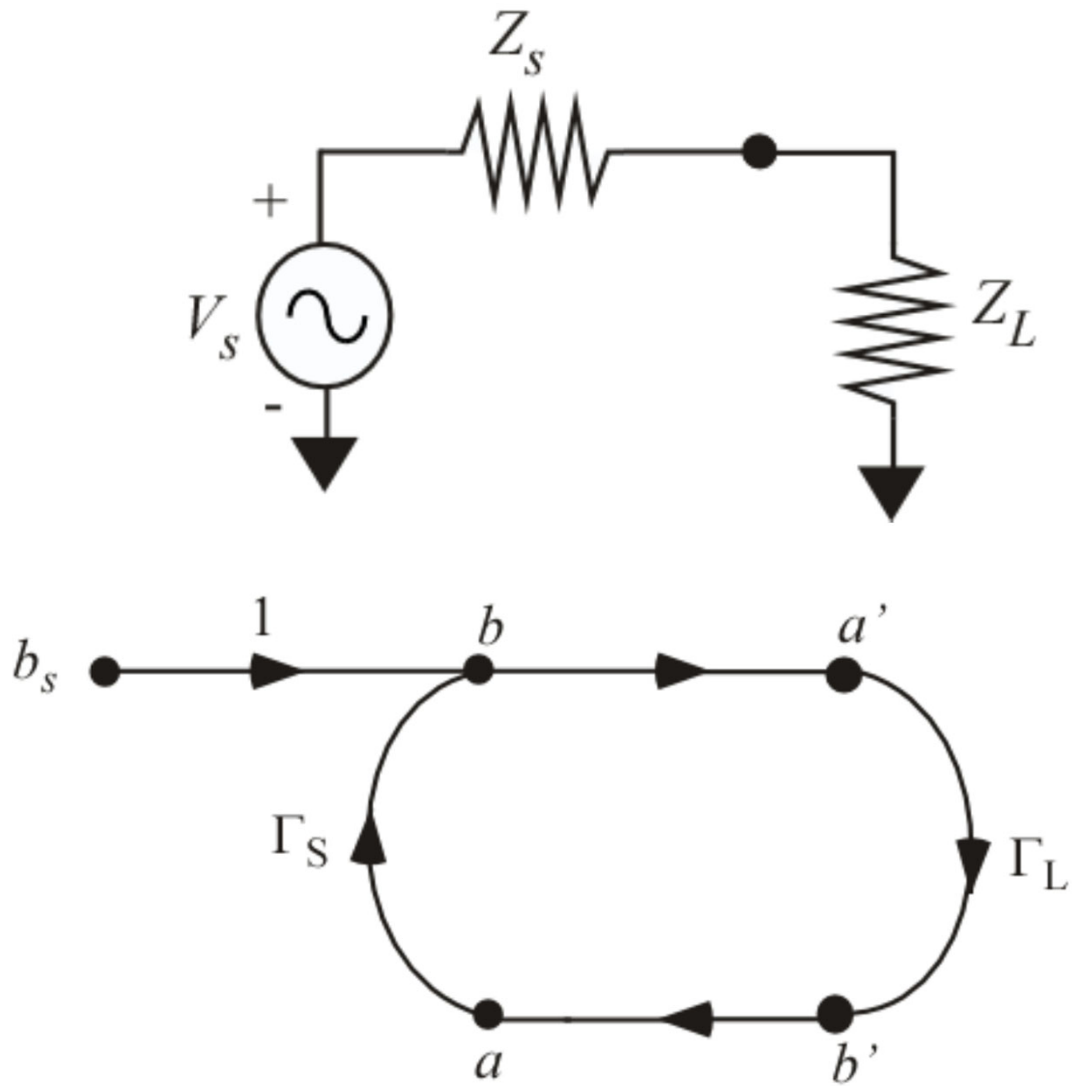
Flow Graph for Load



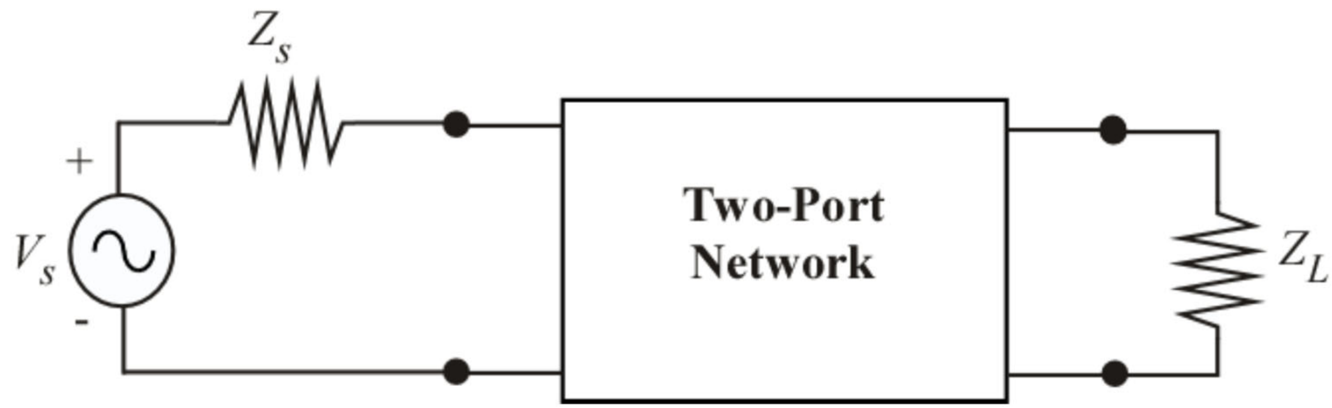


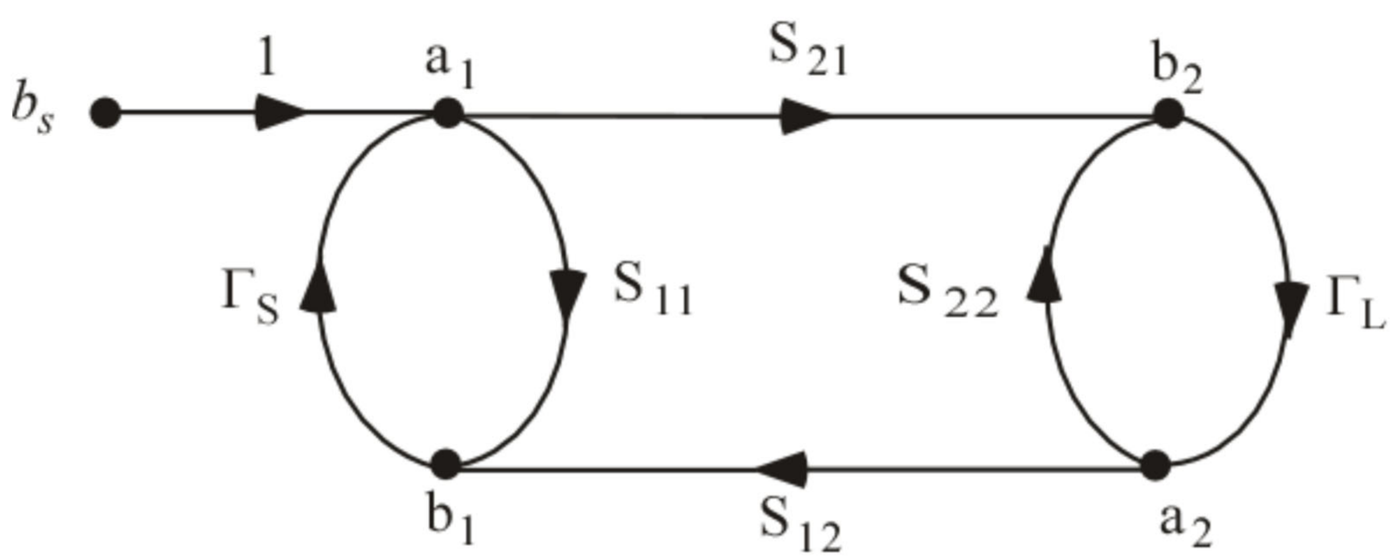
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

Flow Graph for Composite Circuit



Flow Graph of Complete Two-Port





Loop Definitions

- A first order loop is defined as the product of the branches encountered in a journey starting from anode and moving in the direction of the arrows back to that original node

- A second order loop is defined as the product of any two non-touching first order loops.

- A third order loop is defined as the product of any three non-touching first order loops.

Mason's Non-Touching Loop Rule

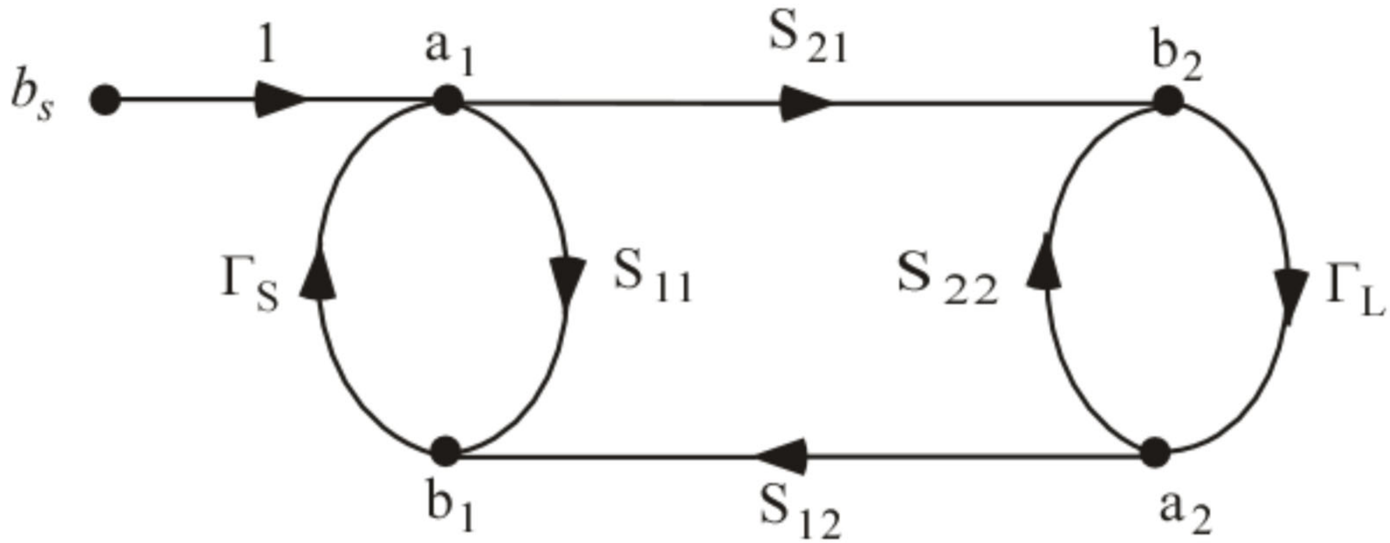
$$T = \frac{P_1\left[1-\sum L(1)^{(1)}+\sum L(2)^{(1)}-\ldots\right]+P_2\left[1-\sum L(1)^{(2)}+\ldots\right]+\ldots}{1-\sum L(1)+\sum L(2)-\sum L(3)+\ldots}$$

T: ratio of dependent variable over independent variable

P_k's: are the various paths connecting the two variables of interest

$L(j)^{(k)}$ is a loop of order j that does not touch path k

Example: Find b_2/b_s



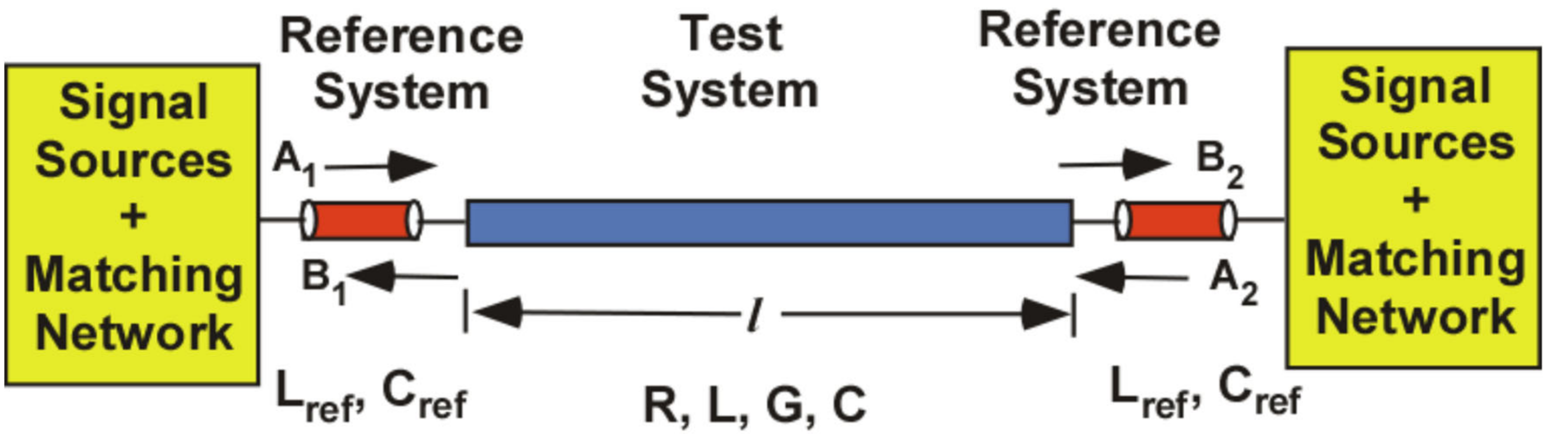
First Order Loops: $S_{11}\Gamma_S, \quad S_{22}\Gamma_L, \quad S_{21}S_{12}\Gamma_L\Gamma_S$

Second Order Loops: $S_{11}\Gamma_S S_{22}\Gamma_L$

Paths: S_{21}

$$\frac{b_2}{b_s} = \frac{S_{21}}{1 - S_{11}\Gamma_S - S_{22}\Gamma_L - S_{21}S_{12}\Gamma_L\Gamma_S + S_{11}\Gamma_S S_{22}\Gamma_L}$$

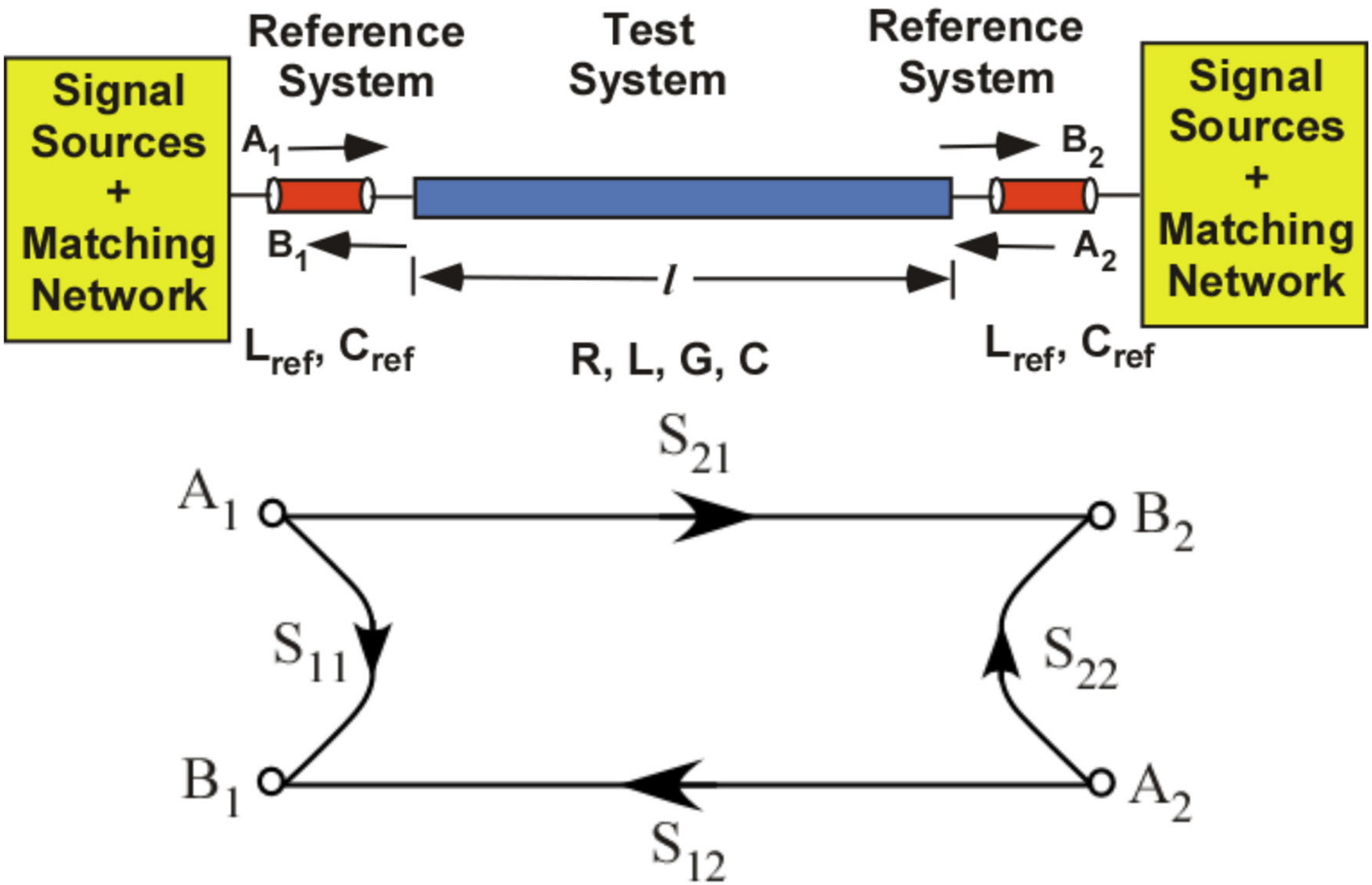
Lossy and Dispersive Line



$$S_{11} = S_{22} = \frac{\left(1-\alpha^2\right)\rho}{1-\rho^2\alpha^2} \qquad S_{21} = S_{12} = \frac{\left(1-\rho^2\right)\alpha}{1-\rho^2\alpha^2}$$

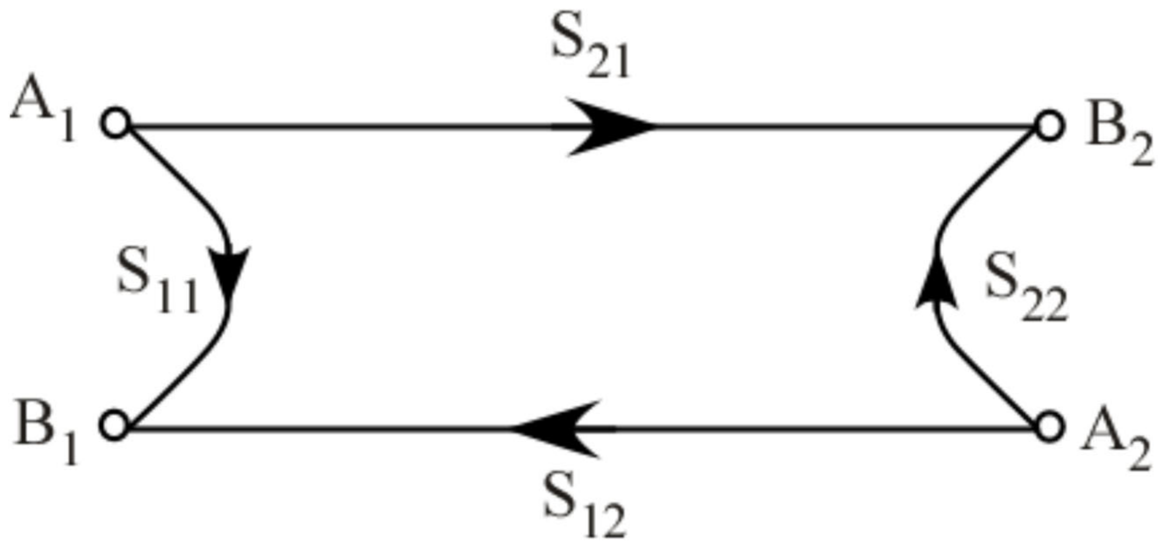
$$\alpha = e^{-\gamma l} \qquad \rho = \frac{Z_c(\omega) - Z_o}{Z_c(\omega) + Z_o}$$

Frequency-Domain Formulation*



* J. E. Schutt-Aine and R. Mittra, "Scattering Parameter Transient analysis of transmission lines loaded with nonlinear terminations," IEEE Trans. Microwave Theory Tech., vol. MTT-36, pp. 529-536, March 1988.

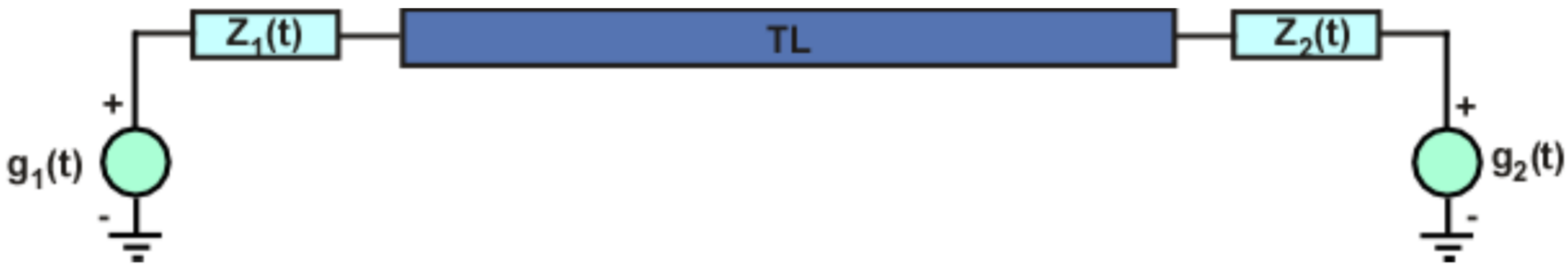
Frequency-Domain

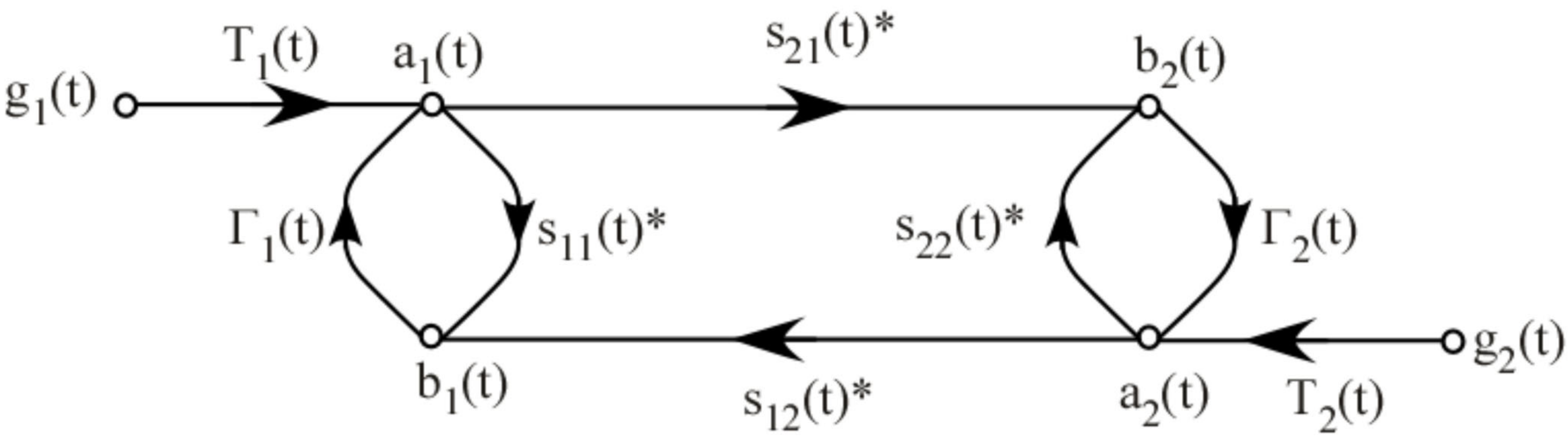


$$B_1(\omega) = S_{11}(\omega) A_1(\omega) + S_{12}(\omega) A_2(\omega)$$

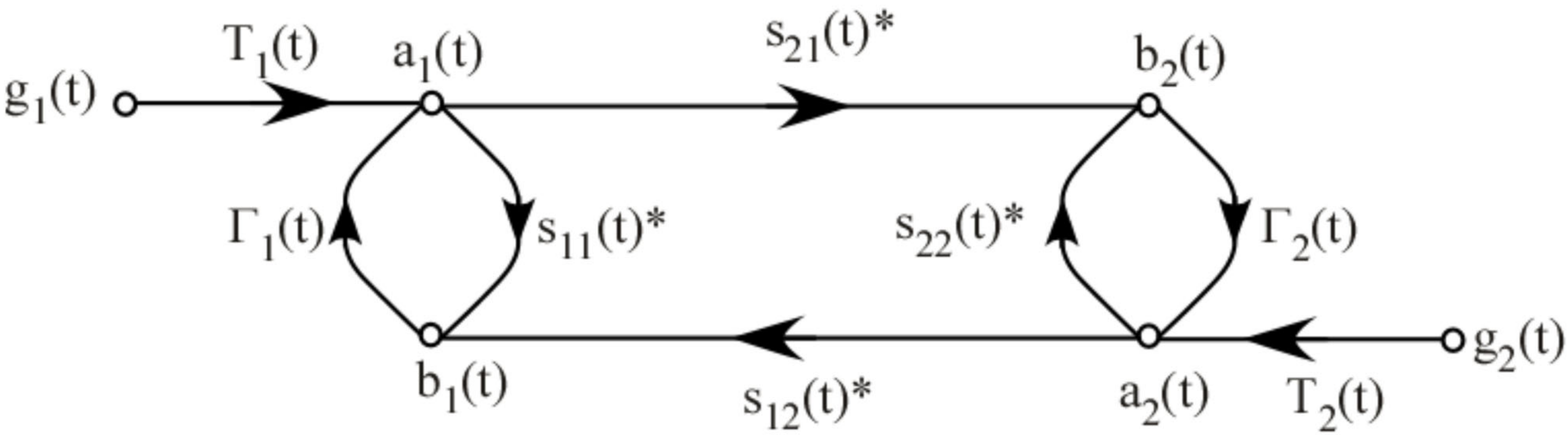
$$B_2(\omega) = S_{21}(\omega) A_1(\omega) + S_{22}(\omega) A_2(\omega)$$

Time-Domain Formulation





Time-Domain Formulation



$$b_1(t) = s_{11}(t) * a_1(t) + s_{12}(t) * a_2(t)$$

$$b_2(t) = s_{21}(t) * a_1(t) + s_{22}(t) * a_2(t)$$

$$T_i(t) = \frac{Z_o}{Z_i(t) + Z_o}$$

$$a_1(t) = \Gamma_1(t) b_1(t) + T_1(t) g_1(t)$$

$$a_2(t) = \Gamma_2(t) b_2(t) + T_2(t) g_2(t)$$

$$\Gamma_i(t) = \frac{Z_i(t) - Z_o}{Z_i(t) + Z_o}$$

Time-Domain Solutions

$$a_1(t) = \frac{\left[1-\Gamma_2(t)s_{22}^{'}(0)\right]\left[T_1(t)g_1(t)+\Gamma_1(t)M_1(t)\right]}{\Delta(t)}$$

$$+\frac{\left[\Gamma_1(t)s_{12}^{'}(0)\right]\left[T_2(t)g_2(t)+\Gamma_2(t)M_2(t)\right]}{\Delta(t)}$$

$$a_2(t) = \frac{\left[1-\Gamma_1(t)s_{11}^{'}(0)\right]\left[T_2(t)g_2(t)+\Gamma_2(t)M_2(t)\right]}{\Delta(t)}$$

$$+\frac{\left[\Gamma_2(t)s_{21}^{'}(0)\right]\left[T_1(t)g_1(t)+\Gamma_1(t)M_1(t)\right]}{\Delta(t)}$$

Time-Domain Solutions

$$b_1(t) = s'_{11}(0)a_1(t) + s'_{12}(0)a_2(t) + M_1(t)$$

$$b_2(t) = s'_{21}(0)a_1(t) + s'_{22}(0)a_2(t) + M_2(t)$$

$$\Delta(t) = \left[1 - \Gamma_1(t)s'_{11}(0)\right]\left[1 - \Gamma_2(t)s'_{22}(0)\right] - \Gamma_1(t)s'_{12}(0)\Gamma_2(t)s'_{21}(0)$$

$$M_1(t) = H_{11}(t) + H_{12}(t)$$

$$M_2(t) = H_{21}(t) + H_{22}(t)$$

$$s'_{ij}(0) = s_{ij}(0)\Delta\tau$$

$$H_{ij}(t) = \sum_{\tau=1}^{t-1} s_{ij}(t-\tau)a_j(\tau)\Delta\tau$$

Special Case – Lossless Line

$$s_{11}(t) = s_{22}(t) = 0 \qquad s_{12}(t) = s_{21}(t) = \delta\left(t - \frac{l}{v}\right)$$

$$M_1(t) = a_2\left(t - \frac{l}{v}\right) \qquad M_2(t) = a_1\left(t - \frac{l}{v}\right)$$

$$a_1(t) = T_1(t)g_1(t) + \Gamma_1(t)a_2\left(t - \frac{l}{v}\right)$$

$$a_2(t) = T_2(t)g_2(t) + \Gamma_2(t)a_1\left(t - \frac{l}{v}\right)$$

$$b_1(t) = a_2\left(t - \frac{l}{v}\right)$$

$$b_2(t) = a_1\left(t - \frac{l}{v}\right)$$

Wave Shifting Solution

Time-Domain Solutions

$$v_1(t) = a_1(t) + b_1(t)$$

$$v_2(t) = a_2(t) + b_2(t)$$

$$i_1(t) = \frac{a_1(t)}{Z_o} - \frac{b_1(t)}{Z_o}$$

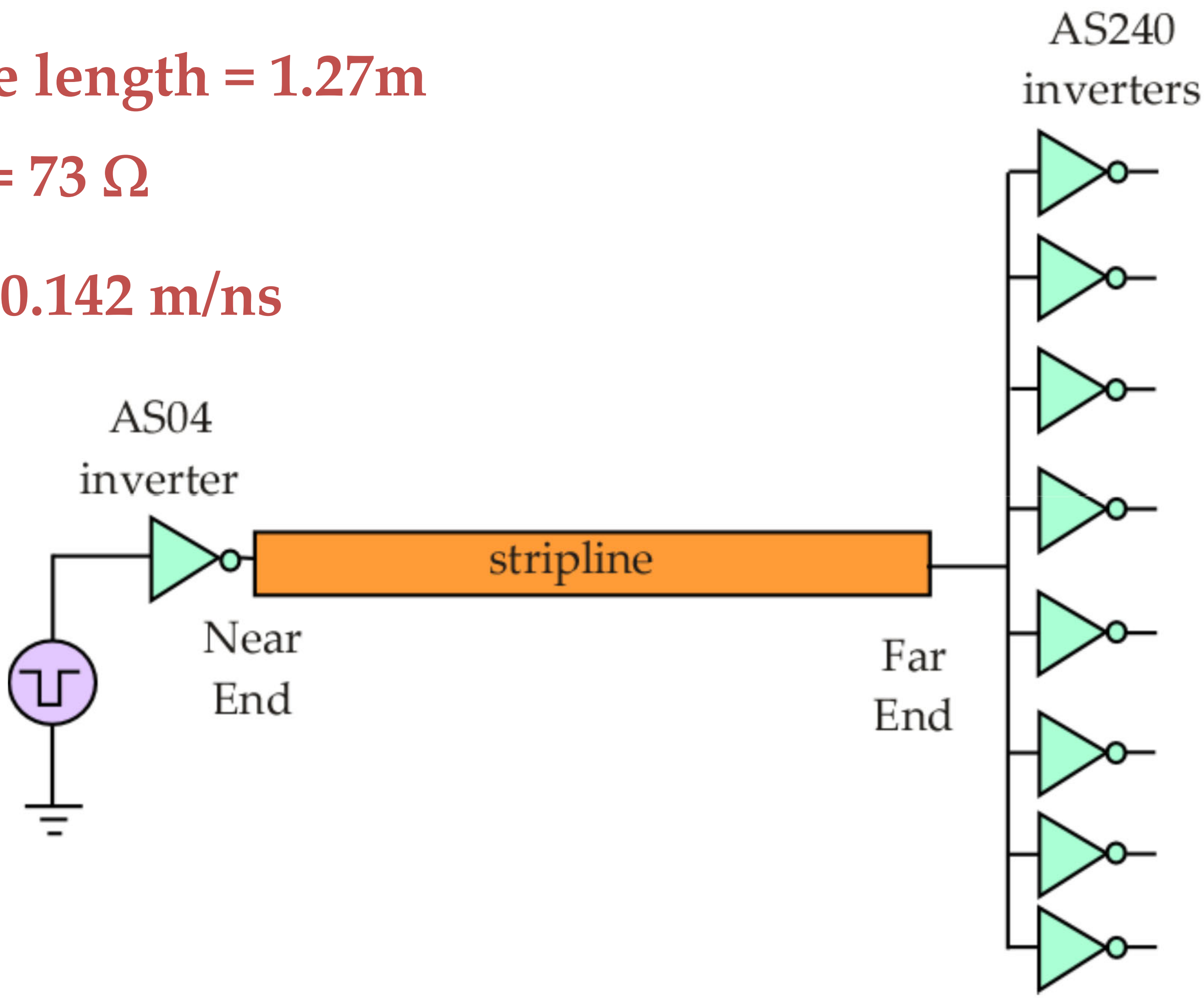
$$i_2(t) = \frac{a_2(t)}{Z_o} - \frac{b_2(t)}{Z_o}$$

Simulations

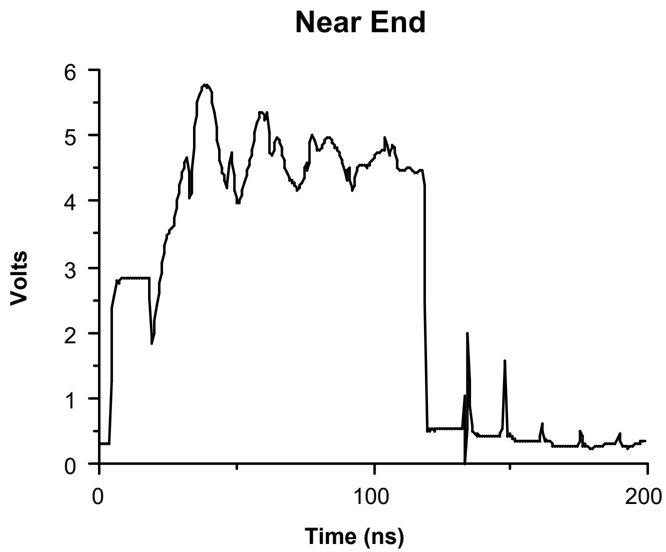
Line length = 1.27m

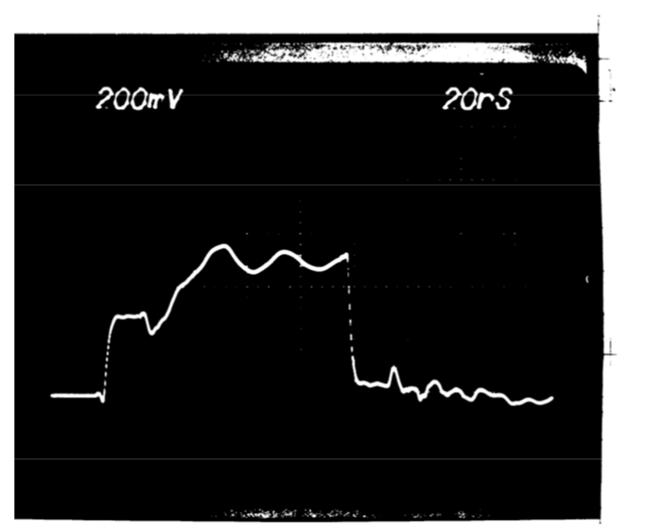
$Z_o = 73\ \Omega$

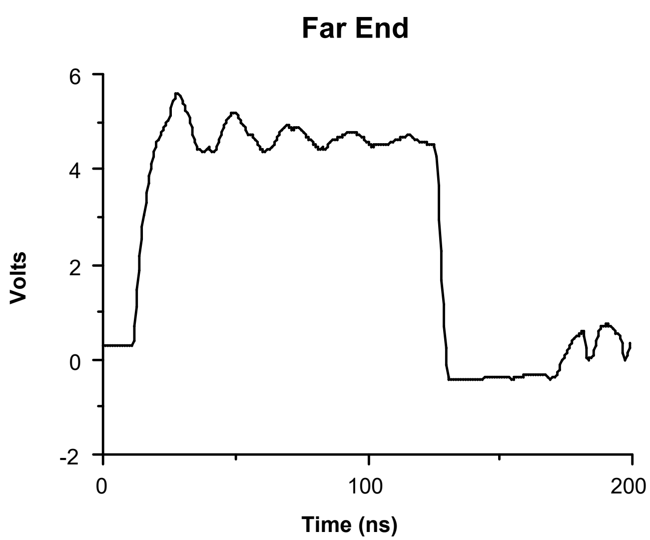
v = 0.142 m/ns

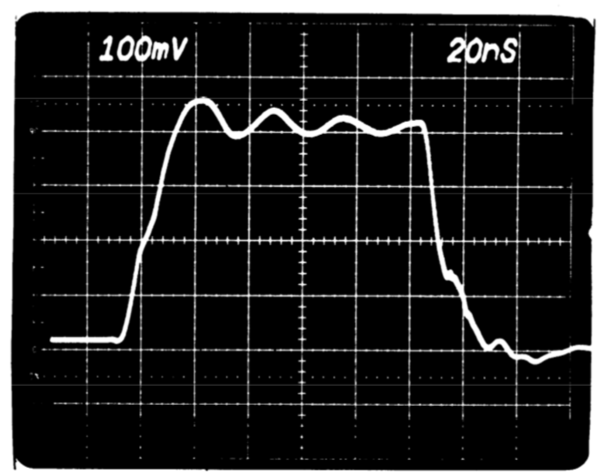


Simulations

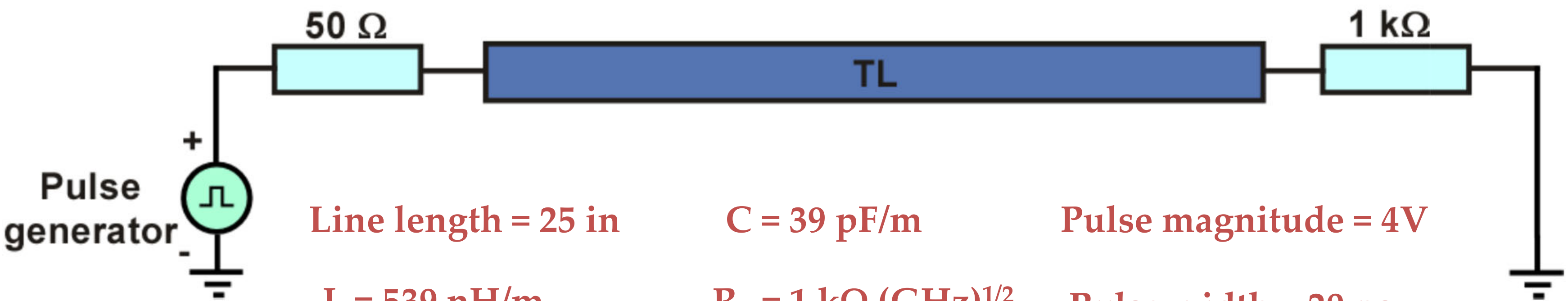








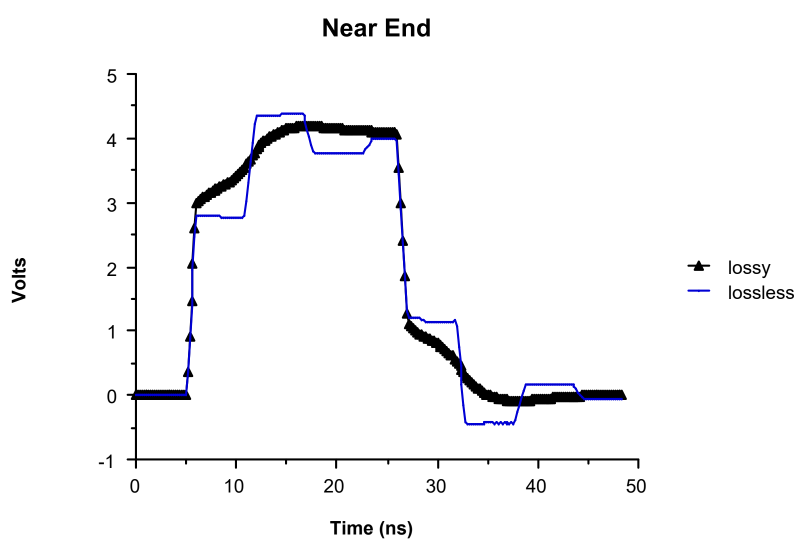
Simulations

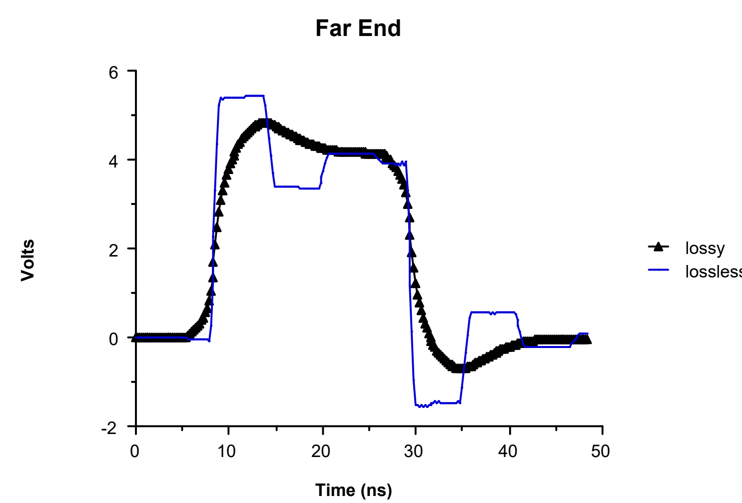


Line length = 25 in   C = 39 pF/m   Pulse magnitude = 4V

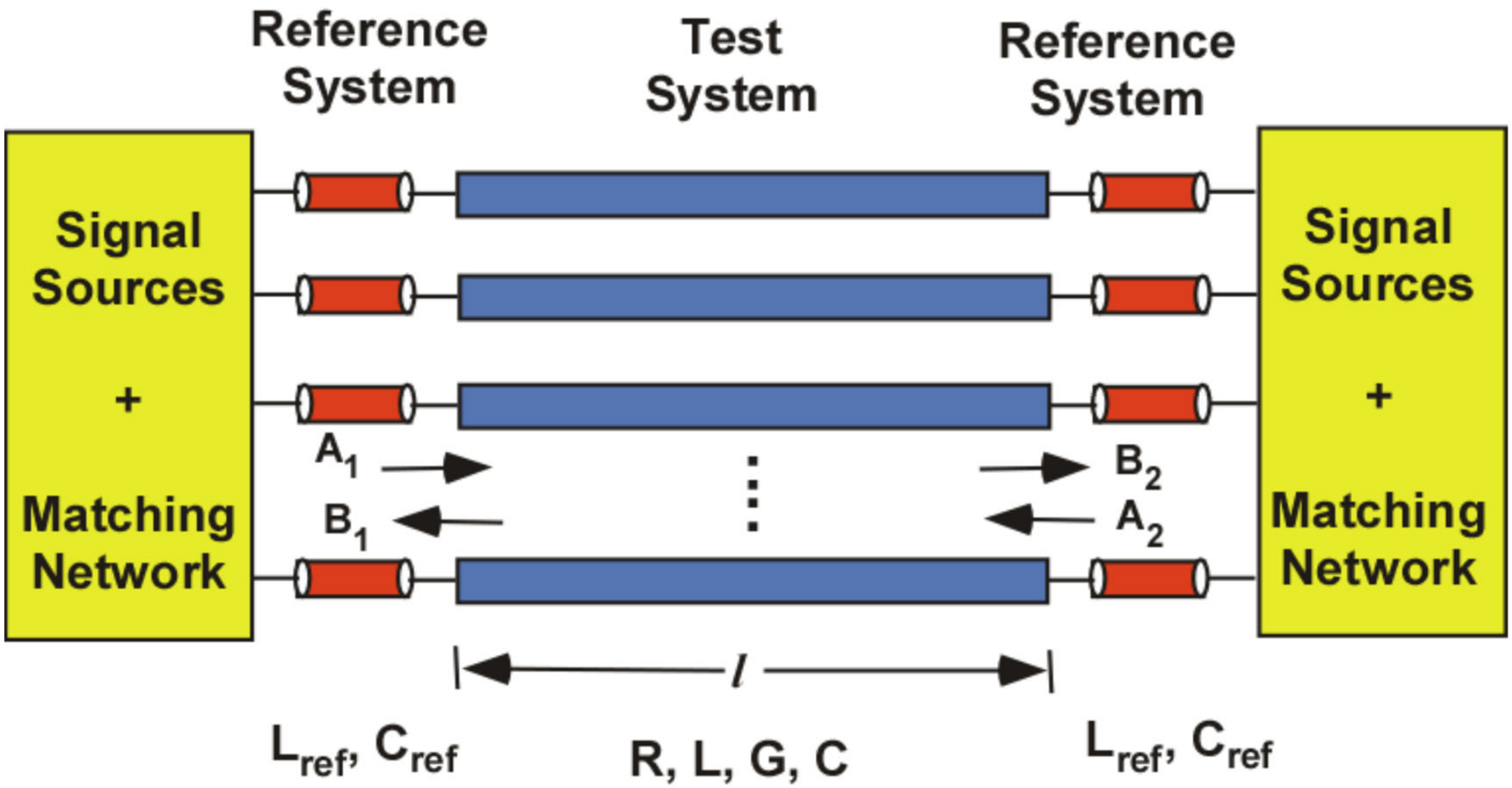
L = 539 nH/m $R_o = 1\ k\Omega\ (GHz)^{1/2}$ Pulse width = 20 ns

Rise and fall times = 1ns





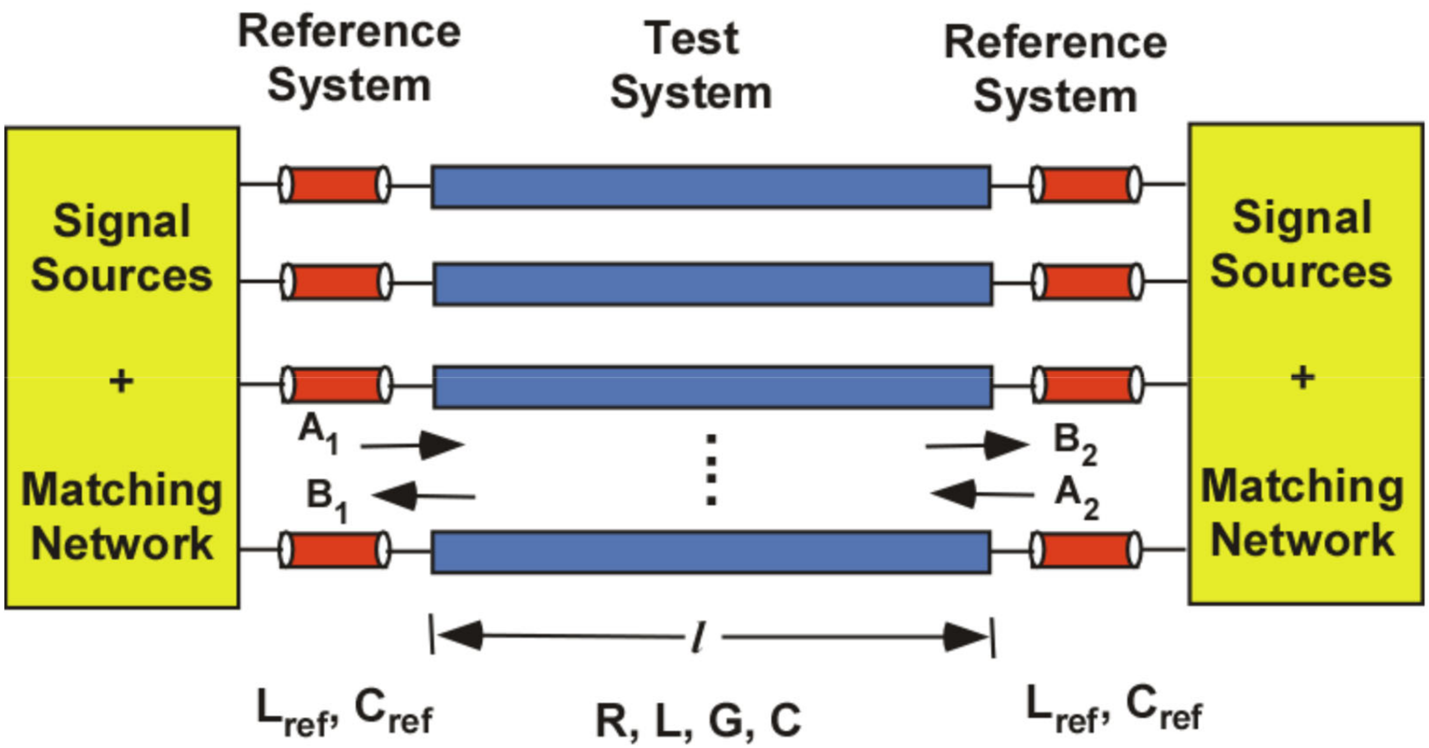
N-Line S-Parameters*



$$B_1 = S_{11} A_1 + S_{12} A_2 \qquad B_2 = S_{21} A_1 + S_{22} A_2$$

* J. E. Schutt-Aine and R. Mittra, "Transient analysis of coupled lossy transmission lines with nonlinear terminations," IEEE Trans. Circuit Syst., vol. CAS-36, pp. 959-967, July 1989.

Scattering Parameters for N-Line



$$S_{21} = S_{12} = 2E_oE^{-1}\left[1-\Gamma\right]\Psi\left[1-\Gamma\Psi\Gamma\Psi\right]^{-1}T$$

$$S_{11} = S_{22} = T^{-1}\left[\Gamma-\Psi\Gamma\Psi\right]\left[1-\Gamma\Psi\Gamma\Psi\right]^{-1}T$$

$$\Gamma = \left[1+EE_o^{-1}Z_oH_oH^{-1}Z_m^{-1}\right]^{-1}\left[1-EE_o^{-1}Z_oH_oH^{-1}Z_m^{-1}\right]$$

$$T = \left[1+EE_o^{-1}Z_oH_oH^{-1}Z_m^{-1}\right]^{-1}EE_o^{-1} \qquad \Psi = W(-l)$$

Scattering Parameter Matrices

$\mathbf{E_o}$: Reference system voltage eigenvector matrix

$\mathbf{E}$: Test system voltage eigenvector matrix

$\mathbf{H_o}$: Reference system current eigenvector matrix

$\mathbf{H}$: Test system current eigenvector matrix

$\mathbf{Z_o}$: Reference system modal impedance matrix

$\mathbf{Z_m}$: Test system modal impedance matrix

Eigen Analysis

* Diagonalize ZY and YZ and find eigenvalues.
* Eigenvalues are complex: $\lambda_i = \alpha_i + j\beta_i$

$$\mathbf{W(u)} = \begin{bmatrix} e^{\alpha_1 u + j\beta_1 u} & & & \\ & e^{\alpha_2 u + j\beta_2 u} & & \\ & & \bullet & \\ & & & e^{\alpha_n u + j\beta_n u} \end{bmatrix}$$

Solution

$$\mathbf{V_m} = \mathbf{EV}$$

$$\mathbf{I_m} = \mathbf{HI}$$

$$\mathbf{V_m}(x) = \left[\mathbf{W}(-x)\mathbf{A} + \mathbf{W}(x)\mathrm{B}\right]$$

$$\mathbf{I_m}(x) = \mathbf{Z_m^{-1}}\left[\mathbf{W}(-x)\mathbf{A} + \mathbf{W}(x)\mathrm{B}\right]$$

$$\mathbf{Z_m} = \mathbf{\Lambda_m^{-1}EZH^{-1}}$$

$$\mathbf{Z_c} = \mathbf{E^{-1}Z_m H} = \mathbf{E^{-1}\Lambda_m^{-1}EZ}$$

Solutions

$$a_1(t) = \Delta_1^{-1}\left[1-\Gamma_1(t)s'_{11}(0)\right]^{-1}\left[T_1(t)g_1(t)+\Gamma_1(t)M_1(t)\right]$$

$$-\Delta_1^{-1}\left[1-\Gamma_1(t)s'_{11}(0)\right]^{-1}\left[1-\Gamma_2(t)s'_{22}(0)\right]^{-1}\times$$

$$\left[\Gamma_1(t)s'_{21}(0)\right]\left[T_2(t)g_2(t)+\Gamma_2(t)M_2(t)\right]$$

$$a_2(t) = \Delta_2^{-1}\left[1-\Gamma_2(t)s'_{22}(0)\right]^{-1}\left[T_2(t)g_2(t)+\Gamma_2(t)M_2(t)\right]$$

$$-\Delta_2^{-1}\left[1-\Gamma_2(t)s'_{22}(0)\right]^{-1}\left[1-\Gamma_1(t)s'_{11}(0)\right]^{-1}\times$$

$$\left[\Gamma_1(t)s'_{12}(0)\right]\left[T_1(t)g_1(t)+\Gamma_1(t)M_1(t)\right]$$

Solutions

$$\Delta_1(t) = 1 - \left[1 - \Gamma_1(t) s'_{11}(0)\right]^{-1} \left[1 - \Gamma_2(t) s'_{22}(0)\right]^{-1} \Gamma_1(t) s'_{21}(0) \Gamma_2(t) s'_{12}(0)$$

$$\Delta_2(t) = 1 - \left[1 - \Gamma_2(t) s'_{22}(0)\right]^{-1} \left[1 - \Gamma_1(t) s'_{11}(0)\right]^{-1} \Gamma_2(t) s'_{12}(0) \Gamma_1(t) s'_{21}(0)$$

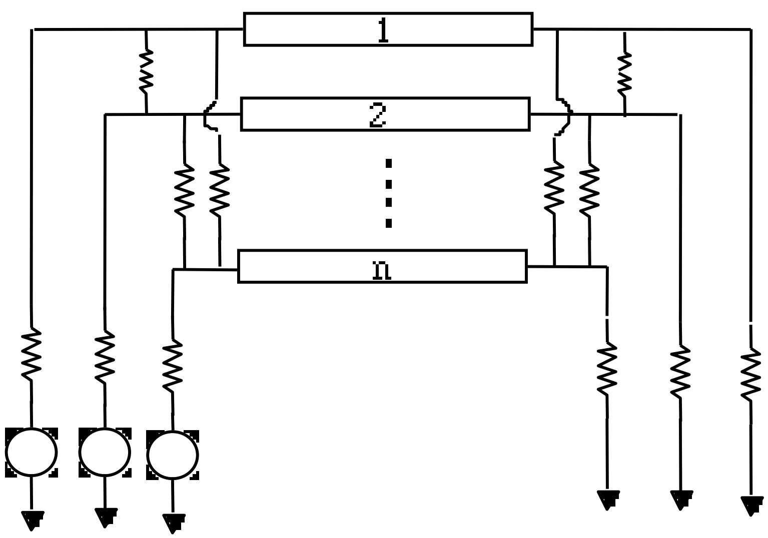
$$b_1(t) = s'_{11}(0) a_1(t) + s'_{12}(0) a_2(t) + M_1(t)$$

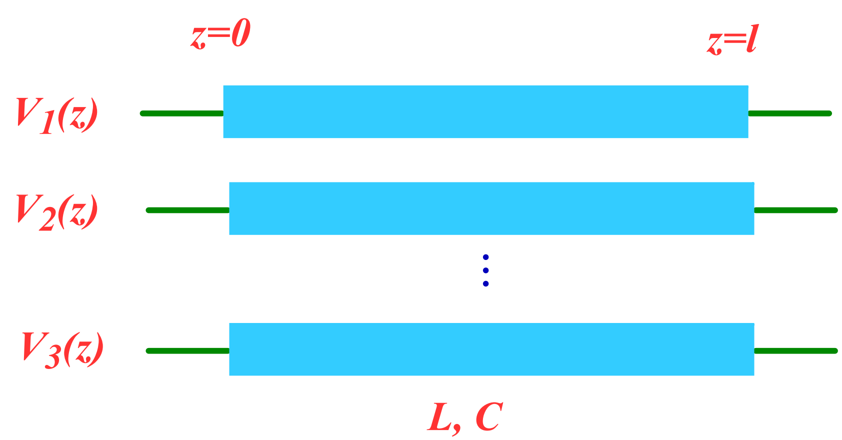
$$b_2(t) = s'_{21}(0) a_1(t) + s'_{22}(0) a_2(t) + M_2(t)$$

Solutions

$$v_{m1}(t) = a_1(t) + b_1(t) \Rightarrow v_1(t) = E_o^{-1}\left[a_1(t) + b_1(t)\right]$$

$$v_{m2}(t) = a_2(t) + b_2(t) \Rightarrow v_2(t) = E_o^{-1}\left[a_2(t) + b_2(t)\right]$$





Lossless Case – Wave Shifting

$$s_{21}(t) = s_{12}(t) = \delta(t - \tau_m)$$

$$M_1(t) = a_2(t - \tau_m)$$

$$M_2(t) = a_1(t - \tau_m)$$

$$a_1(t) = T_1(t) g_1(t) + \Gamma_1(t) a_2(t - \tau_m)$$

$$a_2(t) = T_2(t) g_3(t) + \Gamma_3(t) a_1(t - \tau_m)$$

$$b_1(t) = a_2(t - \tau_m)$$

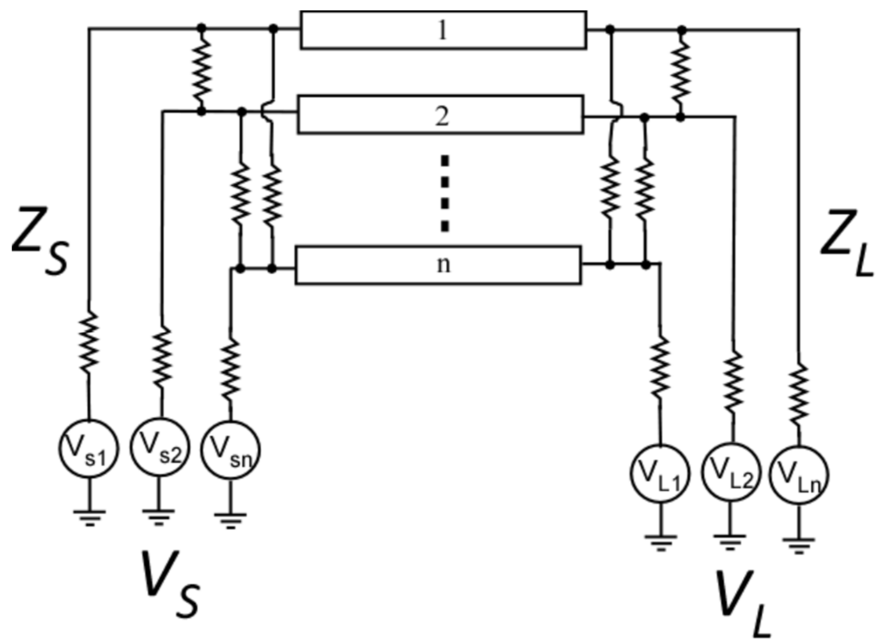
$$b_2(t) = a_2(t - \tau_m)$$

Solution for Lossless Lines

$$\delta(\mathbf{t}-\tau_{\mathbf{m}}) = \begin{pmatrix} \delta(t-\tau_{m1}) & & & \\ & \delta(t-\tau_{m2}) & & \\ & & \bullet & \\ & & & \delta(t-\tau_{mn}) \end{pmatrix}$$

$$\mathbf{a_i}(\mathbf{t}-\tau_{\mathbf{m}}) = \begin{bmatrix} a_1(t-\tau_{m1}) \\ a_2(t-\tau_{m2}) \\ \bullet \\ a_n(t-\tau_{mn}) \end{bmatrix}$$

N-Line Network



Z_s : Source impedance matrix

Z_L : Load impedance matrix

V_s : Source vector

N-Line Simulation

- Get $\boldsymbol{L}$ and $\boldsymbol{C}$ matrices and calculate $\boldsymbol{LC}$ product
- Get square root of eigenvalues and eigenvectors of $\boldsymbol{LC}$ matrix Λ_m
- Arrange eigenvectors into the voltage eigenvector matrix $\boldsymbol{E}$
- Get square root of eigenvalues and eigenvectors of $\boldsymbol{CL}$ matrix Λ_m
- Arrange eigenvectors into the current eigenvector matrix **H**
- Invert matrices $\boldsymbol{E}, \boldsymbol{H}, \Lambda_m$.
- Calculate the line impedance matrix Z_C ➔ $Z_C = E^{-1}\Lambda_m^{-1}EL$
- Construct source and load impedance matrices $Z_s(t)$ and $Z_L(t)$
- Construct source and load reflection coefficient matrices $\Gamma_1(t)$ and $\Gamma_2(t)$. Indices 1 and 2 refer to near and far ends respectively.

$$\Gamma_1(t) = -\left[1 + EZ_sZ_C^{-1}E^{-1}\right]^{-1}\left[1 - EZ_sZ_C^{-1}E^{-1}\right]$$

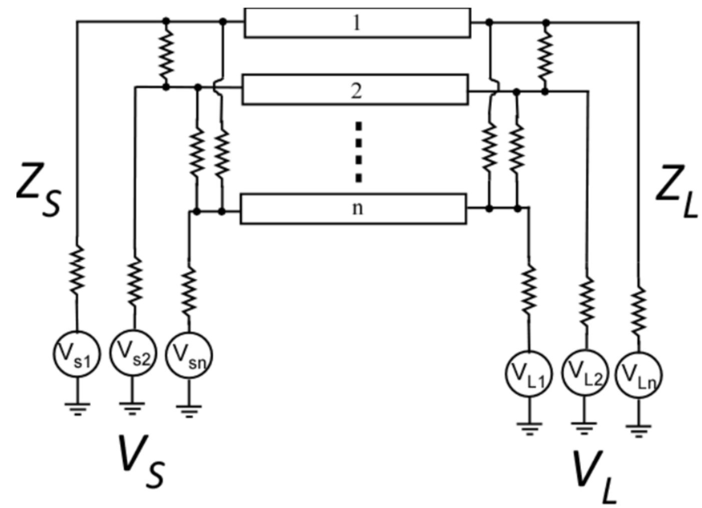
$$\Gamma_2(t) = -\left[1 + EZ_LZ_C^{-1}E^{-1}\right]^{-1}\left[1 - EZ_LZ_C^{-1}E^{-1}\right]$$

Procedure for Coupled Lines

- Construct source and load transmission coefficient matrices $T_1(t)$ and $T_2(t)$

$$T_1(t) = \left[1 + EZ_S Z_C^{-1} E^{-1}\right]^{-1}$$

$$T_2(t) = \left[1 + EZ_L Z_C^{-1} E^{-1}\right]^{-1}$$



- Calculate modal voltage source vectors:

$$g_1(t) = EV_S(t)$$

$$g_2(t) = EV_L(t)$$

$$V_S = \begin{bmatrix} V_{s1} \\ V_{s2} \\ . \\ V_{sn} \end{bmatrix} \qquad V_L = \begin{bmatrix} V_{L1} \\ V_{L2} \\ . \\ V_{Ln} \end{bmatrix}$$

Procedure for Coupled Lines

- Calculate modal voltage wave vectors:

$$\boldsymbol{a_1}(t) = \boldsymbol{T_1}(t)\boldsymbol{g_1}(t) + \boldsymbol{\Gamma_1}(t)\boldsymbol{a_2}(t-\tau_m)$$

$$\boldsymbol{a_2}(t) = \boldsymbol{T_2}(t)\boldsymbol{g_2}(t) + \boldsymbol{\Gamma_2}(t)\boldsymbol{a_1}(t-\tau_m)$$

$$\boldsymbol{b_1}(t) = \boldsymbol{a_2}(t-\tau_m)$$

$$\boldsymbol{b_2}(t) = \boldsymbol{a_2}(t-\tau_m)$$

where $$\boldsymbol{a_2}(t-\tau_m) = \begin{bmatrix} a_{i-\text{mode-1}}(t-\tau_{m1}) \\ a_{i-\text{mode-2}}(t-\tau_{m2}) \\ \cdot \\ a_{i-\text{mode-n}}(t-\tau_{mn}) \end{bmatrix}$$

τ_{mi} is the delay associated with mode *i*. τ_{mi} = length/velocity of mode *i*. The modal voltage wave vectors $\boldsymbol{a_1}$(t) and $\boldsymbol{a_2}$(t) need to be stored since they contain the history of the system.

Procedure for Coupled Lines

- Calculate total modal voltage wave vectors:

$$\mathbf{V}_{m1}(t) = \mathbf{a}_1(t) + \mathbf{b}_1(t)$$

$$\mathbf{V}_{m2}(t) = \mathbf{a}_2(t) + \mathbf{b}_2(t)$$

- Calculate line voltage wave vectors

$$\mathbf{V}_1(t) = \mathbf{E}^{-1}\mathbf{V}_{m1}(t)$$

$$\mathbf{V}_2(t) = \mathbf{E}^{-1}\mathbf{V}_{m2}(t)$$

$$\mathbf{V}_1(t) = \begin{bmatrix} V_{near-line-1} \\ V_{near-line-2} \\ \\ V_{near-line-n} \end{bmatrix} \quad \mathbf{V}_2(t) = \begin{bmatrix} V_{far-line-1} \\ V_{far-line-2} \\ \\ V_{far-line-n} \end{bmatrix}$$

Note: subscripts 1 and 2 refer to near and far ends respectively

N-Line - Additional Notes

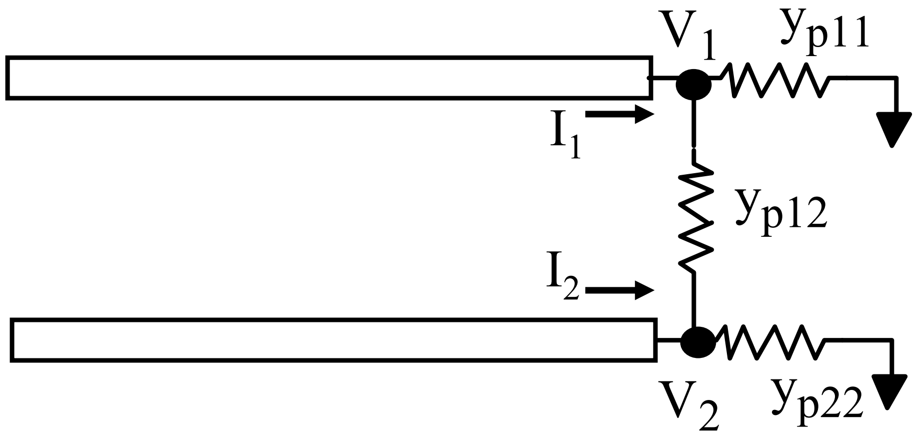
Eigenvalue matrix

$$\Lambda_m = \begin{bmatrix} \frac{1}{v_{m1}} & & & \\ & \frac{1}{v_{m2}} & & \\ & & \bullet & \\ & & & \frac{1}{v_{mn}} \end{bmatrix}$$

For two-line case, two modes: even and odd. In addition, ***E*** and ***H*** are equal and are independent of the entries of ***L*** and ***C***

$$\boldsymbol{E} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Source and Load Terminations



$$I_1 = y_{p11}V_1 + y_{p12}\left(V_1 - V_2\right) \qquad I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{p22}V_2 + y_{p12}\left(V_2 - V_1\right) \qquad I_2 = y_{21}V_1 + y_{22}V_2$$

$$y_{11} = y_{p11} + y_{p12}$$

$$y_{22} = y_{p22} + y_{p12}$$

$$y_{12} = y_{21} = -y_{p12}$$

In general for a multiline system

$$\mathbf{I} = \mathbf{YV} \Rightarrow \mathbf{V} = \mathbf{ZY} \qquad \mathbf{Z} = \mathbf{Y}^{-1}$$

Note: $y_{ii} \neq y_{pii}$

$$y_{ij} = -y_{pij} \quad \text{for } i \neq j$$

$$z_{ij} \neq \frac{1}{y_{pij}}$$

Termination Network Construction

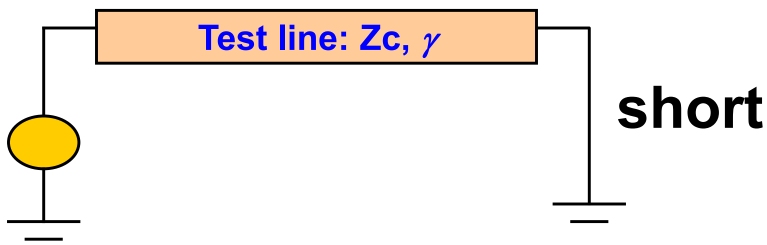
To get impedance matrix Z

- **Get physical impedance values y_{pij}**
- **Calculate y_{ij}'s from y_{pij}'s**
- **Construct Y matrix**
- **Invert Y matrix to obtain Z matrix**

Remark: If $y_{pij} = 0$ for all $i \neq j$, then $Y = Z^{-1}$ and $z_{ii} = 1/y_{ii}$

Why Use S Parameters?

Y-Parameter

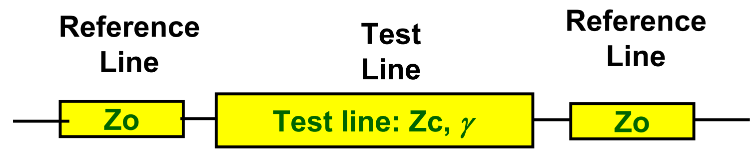


$$Y_{11} = \frac{1+e^{-2\gamma l}}{Z_c(1-e^{-2\gamma l})}$$

Z_c : microstrip characteristic impedance
γ : complex propagation constant
***l* : length of microstrip**

Y_{11} can be unstable

S-Parameter



$$S_{11} = \frac{(1-e^{-2\gamma l})\Gamma}{1-\Gamma^2 e^{-2\gamma l}}$$

$$\Gamma = \frac{Z_c - Z_o}{Z_c + Z_o}$$

S_{11} is always stable

Choice of Reference

$$\Gamma = \frac{Z_c - Z_{ref}}{Z_c + Z_{ref}} \qquad\qquad Z_c = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Z_{ref} is arbitrary

What is the best choice for Z_{ref} ?

At high frequencies $\quad Z_c \rightarrow \sqrt{\frac{L}{C}}$

Thus, if we choose $\quad Z_{ref} = \sqrt{\frac{L}{C}}$

$$S_{12} \rightarrow e^{-j\omega\sqrt{LC}d} = X_o \qquad\qquad S_{11} \rightarrow 0$$

Choice of Reference

S-Parameter measurements (or simulations) are made using a 50-ohm system. For a 4-port, the reference impedance is given by:

$$Z_o = \begin{matrix} 50.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 50.0 \end{matrix}$$

Z: Impedance matrix (of blackbox)
S: S-parameter matrix
Z_o: Reference impedance
I: Unit matrix

$$S = \left[ZZ_o^{-1} + I \right]^{-1} \left[ZZ_o^{-1} - I \right]$$

$$Z = [I + S][I - S]^{-1} Z_o$$

Reference Transformation

<u>Method:</u> Change reference impedance from uncoupled to coupled system to get new S-parameter representation

$$Z_o = \begin{bmatrix} 50.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 50.0 \end{bmatrix}$$

Uncoupled system

$$Z_o = \begin{bmatrix} 328.0 & 69.6 & 328.9 & 69.6 \\ 69.6 & 328.8 & 69.6 & 328.9 \\ 328.9 & 69.6 & 328.8 & 69.6 \\ 69.6 & 328.9 & 69.6 & 328.8 \end{bmatrix}$$

Coupled system

as an example…

Choice of Reference

using

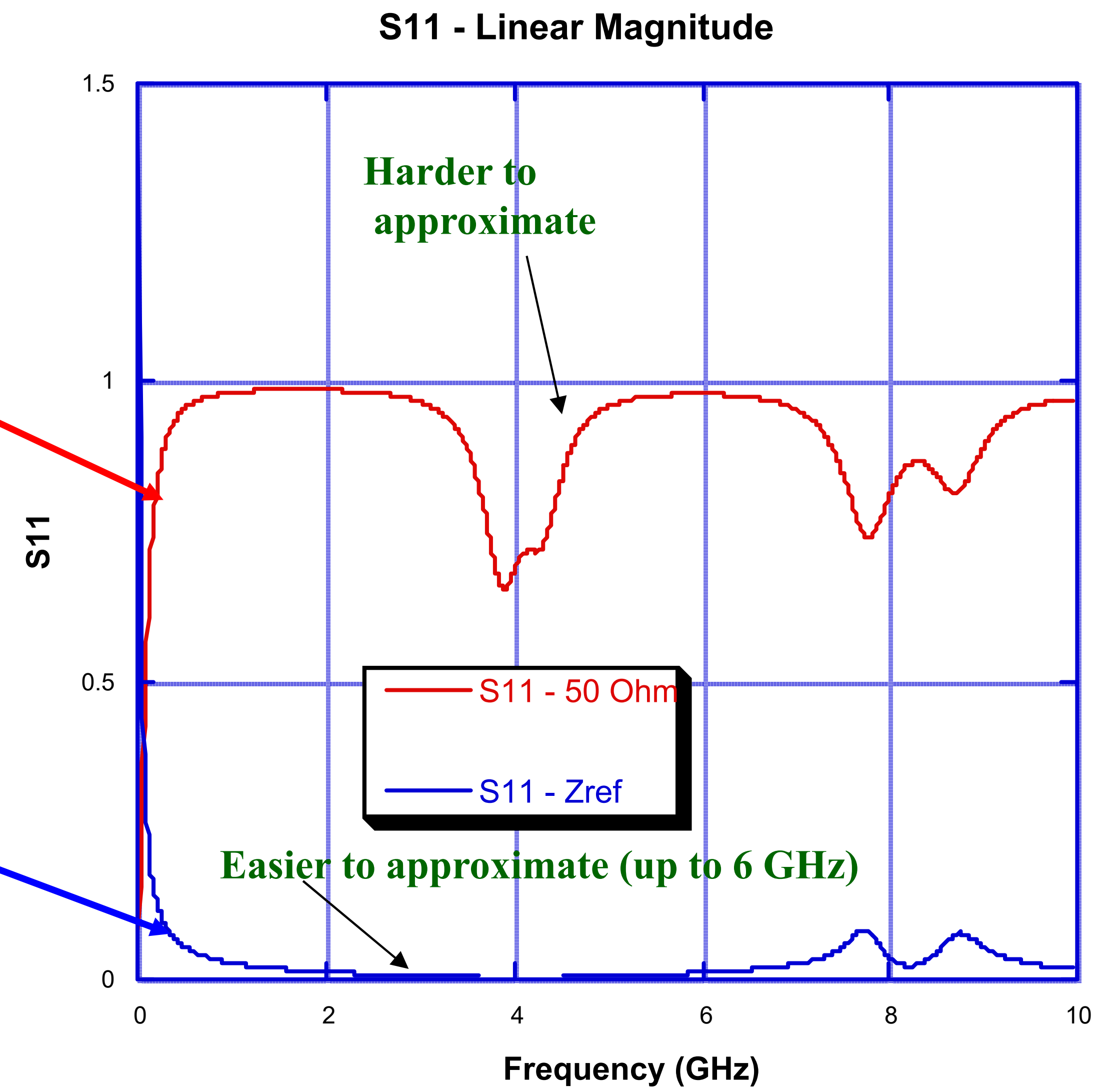
$$Z_o = \begin{matrix} 50.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 50.0 \end{matrix}$$

as reference…

using

$$Z_o = \begin{matrix} 328.0 & 69.6 & 328.9 & 69.6 \\ 69.6 & 328.8 & 69.6 & 328.9 \\ 328.9 & 69.6 & 328.8 & 69.6 \\ 69.6 & 328.9 & 69.6 & 328.8 \end{matrix}$$

as reference…



Choice of Reference

using

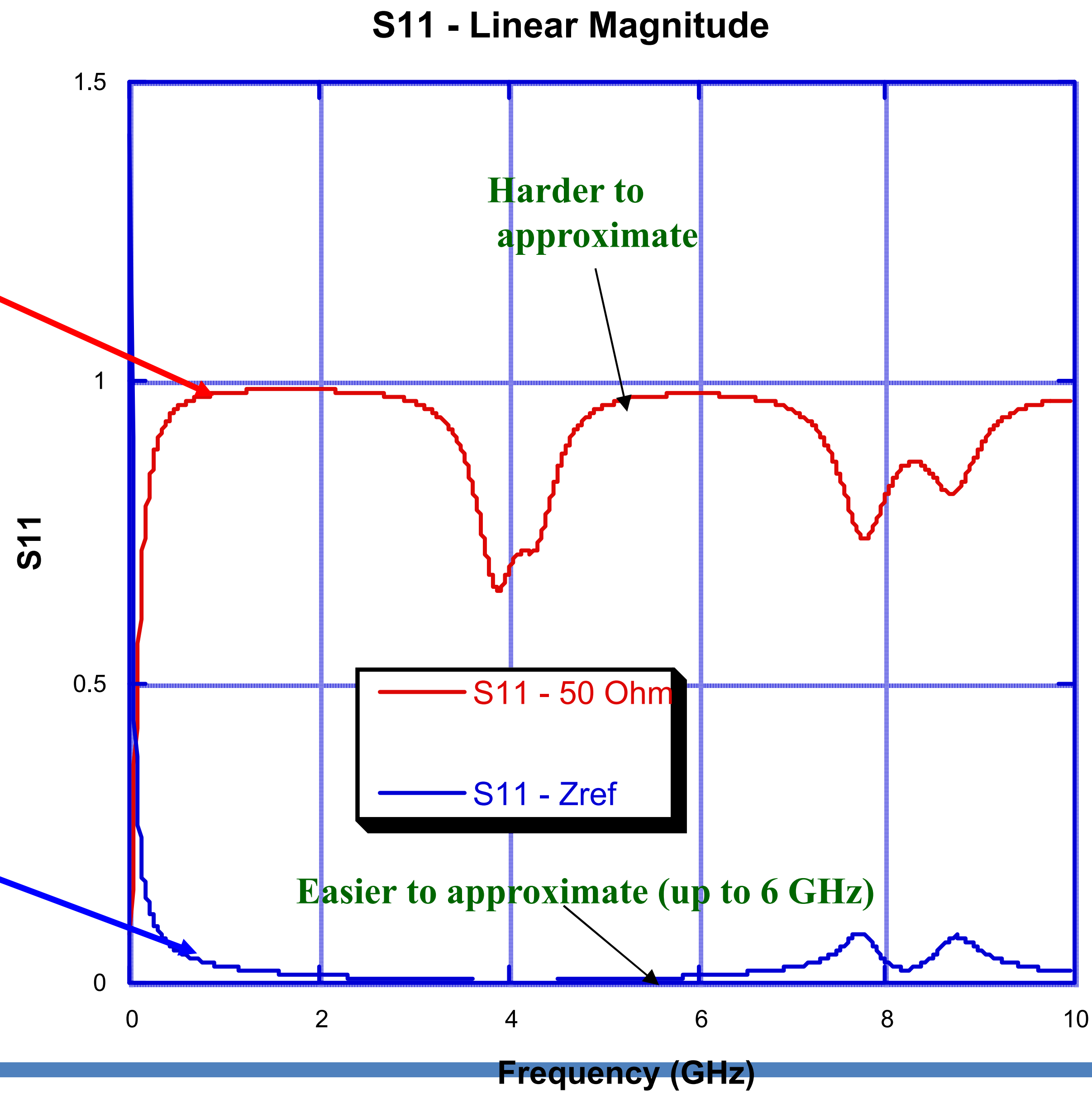
$$Z_o = \begin{matrix} 50.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 50.0 \end{matrix}$$

as reference…

using

$$Z_o = \begin{matrix} 328.0 & 69.6 & 328.9 & 69.6 \\ 69.6 & 328.8 & 69.6 & 328.9 \\ 328.9 & 69.6 & 328.8 & 69.6 \\ 69.6 & 328.9 & 69.6 & 328.8 \end{matrix}$$

as reference…

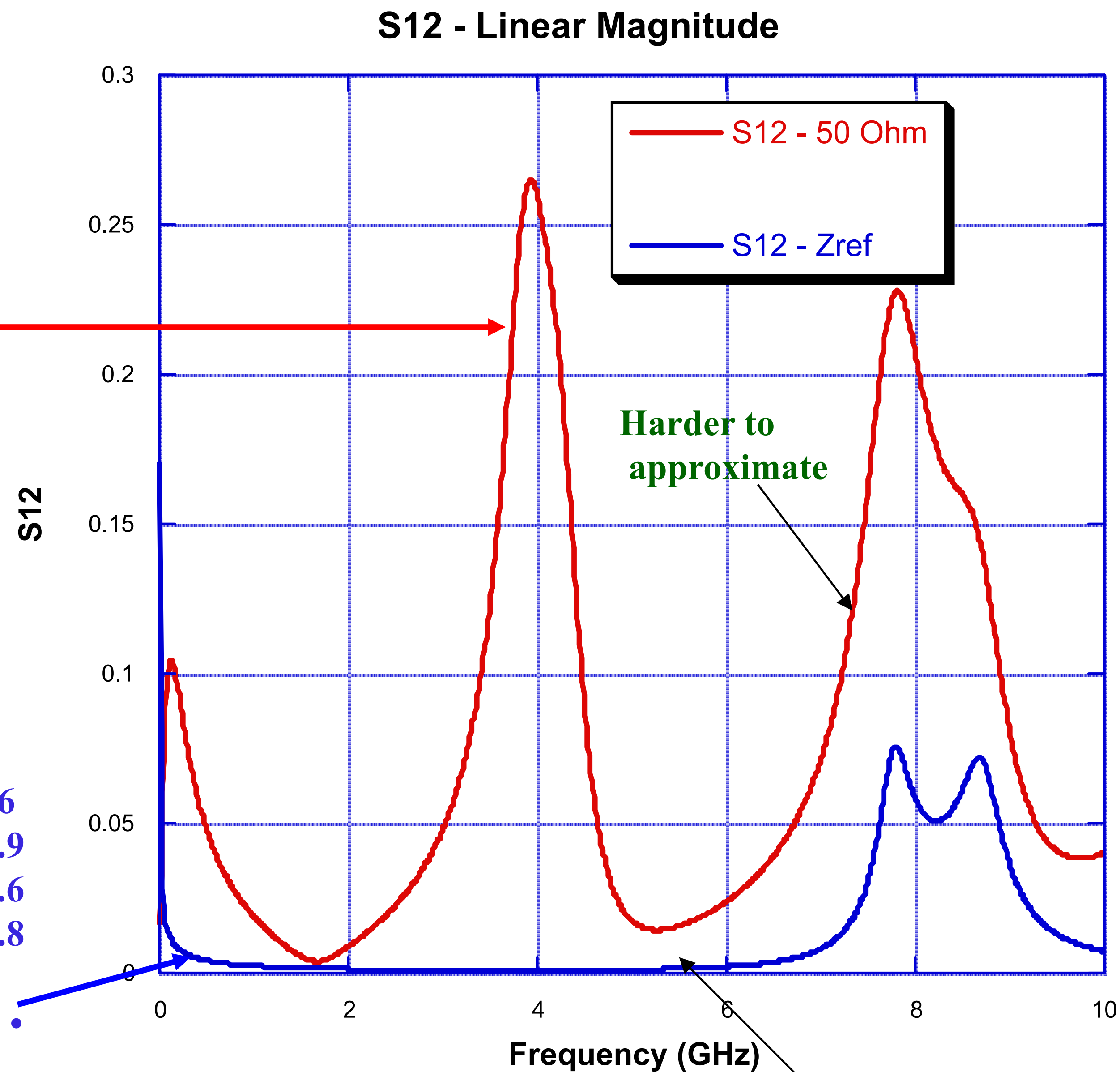


Choice of Reference

using

$$Z_o = \begin{matrix} 50.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 50.0 \end{matrix}$$

as reference…



using

$$Z_o = \begin{matrix} 328.0 & 69.6 & 328.9 & 69.6 \\ 69.6 & 328.8 & 69.6 & 328.9 \\ 328.9 & 69.6 & 328.8 & 69.6 \\ 69.6 & 328.9 & 69.6 & 328.8 \end{matrix}$$

as reference…

Easier to approximate (up to 6 GHz)

Choice of Reference

using

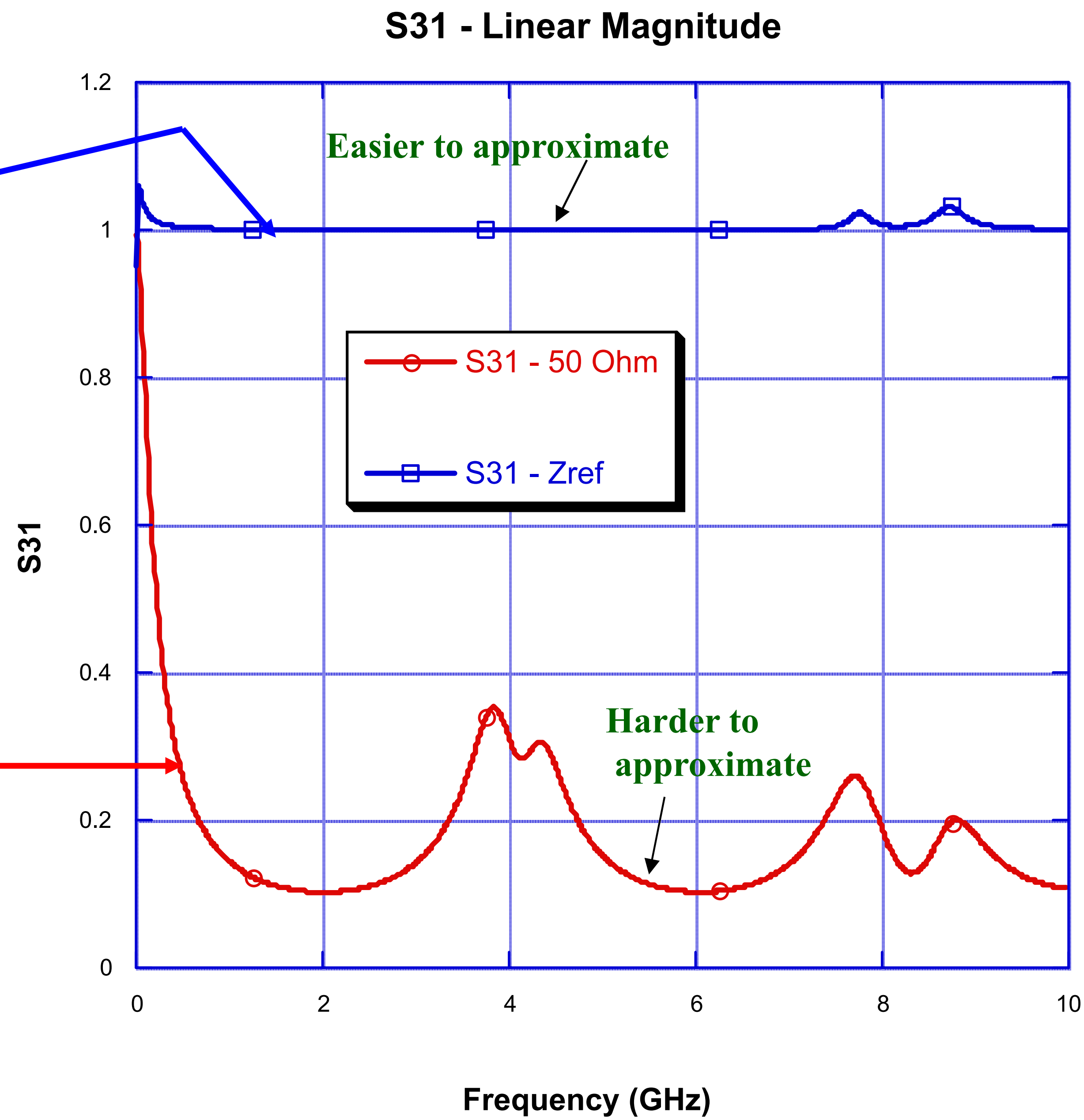
$$Z_o = \begin{matrix} 328.0 & 69.6 & 328.9 & 69.6 \\ 69.6 & 328.8 & 69.6 & 328.9 \\ 328.9 & 69.6 & 328.8 & 69.6 \\ 69.6 & 328.9 & 69.6 & 328.8 \end{matrix}$$

as reference…

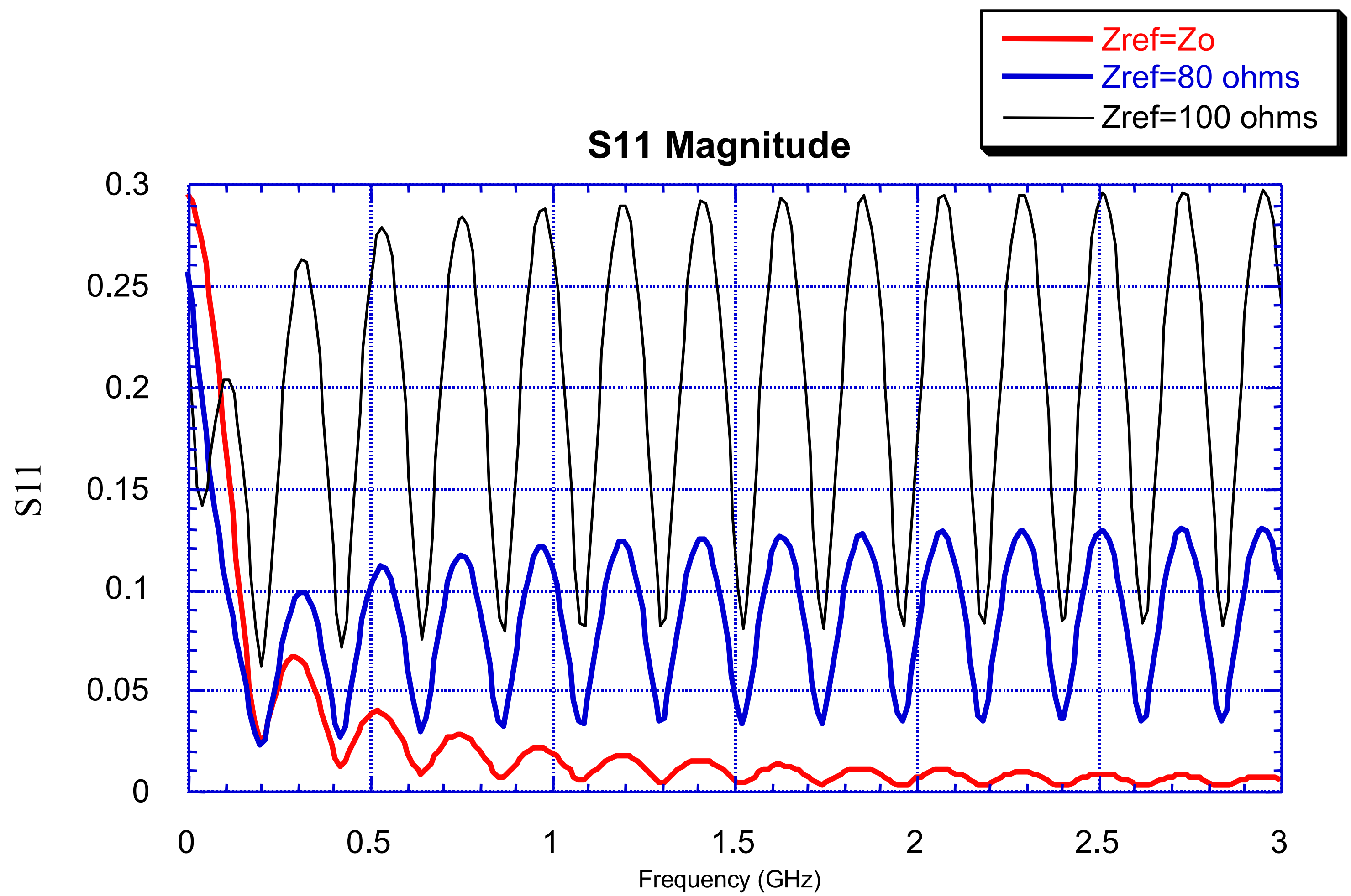
using

$$Z_o = \begin{matrix} 50.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 50.0 \end{matrix}$$

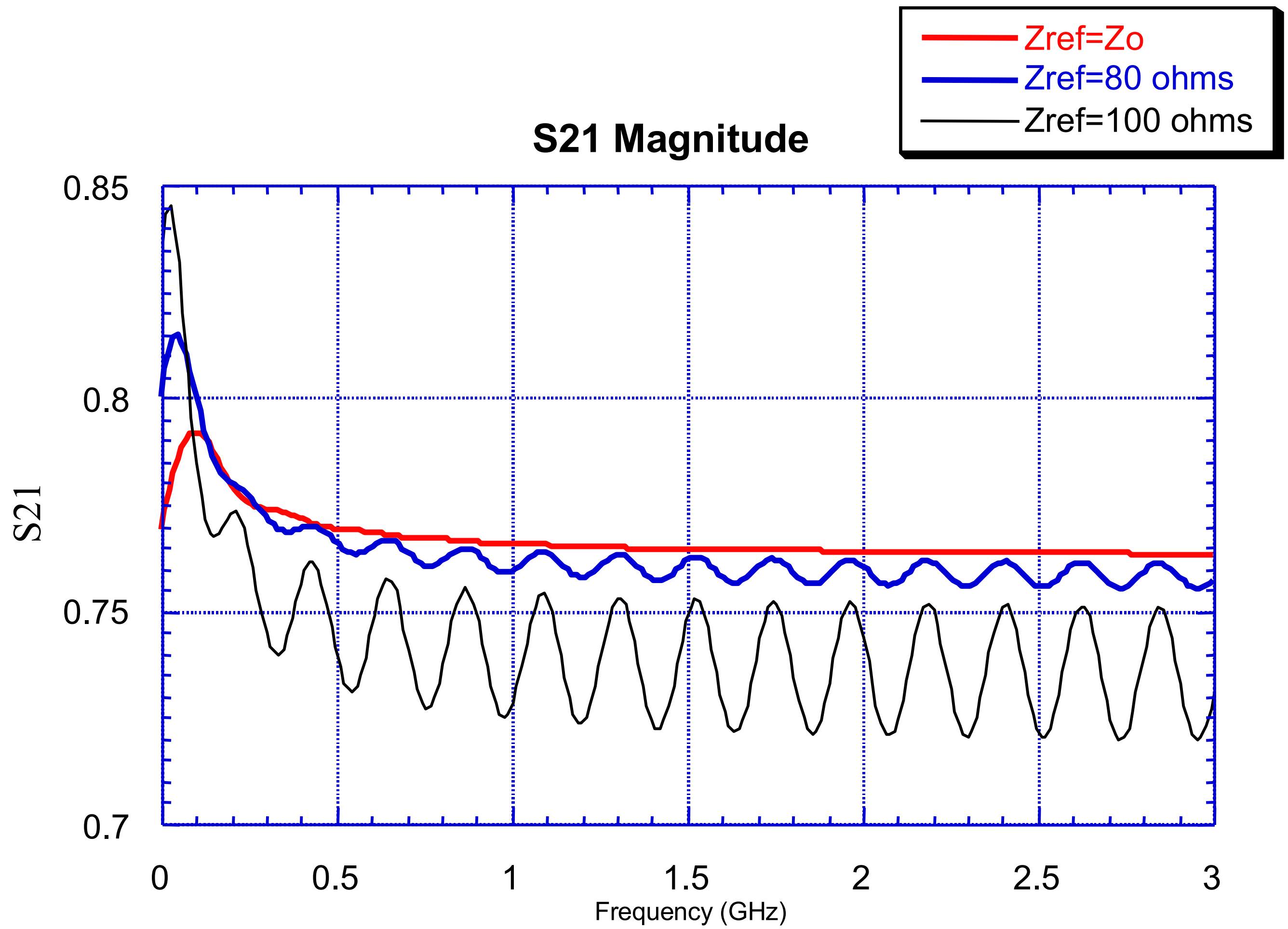
as reference…



Choice of Reference



Choice of Reference



Differential Scattering Parameters

$$S_{AA}=\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

$$S_{AB}=\begin{bmatrix} s_{13} & s_{14} \\ s_{23} & s_{24} \end{bmatrix}$$

$$S_{BA}=\begin{bmatrix} s_{31} & s_{32} \\ s_{41} & s_{42} \end{bmatrix}$$

$$S_{BB}=\begin{bmatrix} s_{33} & s_{34} \\ s_{43} & s_{44} \end{bmatrix}$$

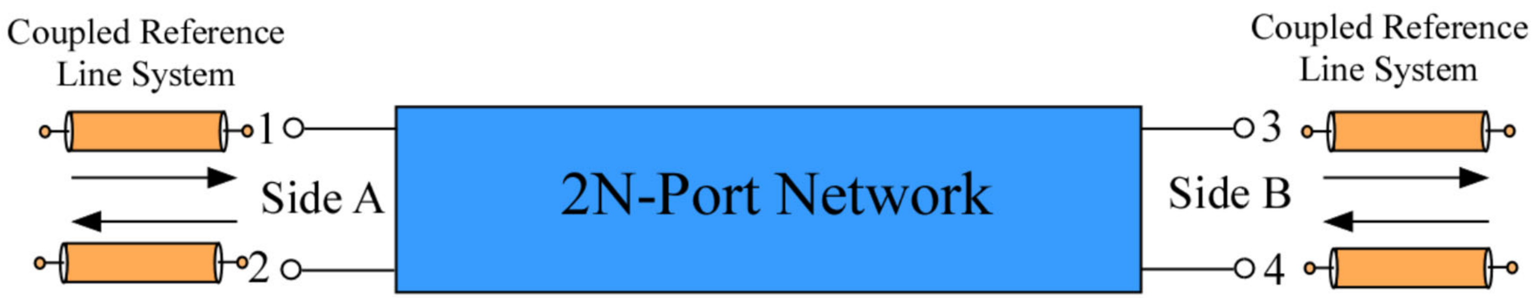
Differential Scattering Parameters

$$S_{Modal} = E_o S_{Line} E_o^{-1}$$

Where E_o is the eigenvector matrix associated with the reference line.

$$E_o = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad E_o^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Differential Scattering Parameters



$$S_{AA-Modal} = E_o S_{AA} E_o^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$S_{AA-Modal} = \begin{bmatrix} \overbrace{(s_{11}+s_{12})+(s_{21}+s_{22})}^{common-to-common} & \overbrace{(s_{11}-s_{12})+(s_{21}-s_{22})}^{common-to-differential} \\ \underbrace{(s_{11}+s_{12})-(s_{21}+s_{22})}_{differential-to-common} & \underbrace{(s_{11}-s_{12})-(s_{21}-s_{22})}_{differential-to-differential} \end{bmatrix}$$

Differential Scattering Parameters

$$S_{BA-Modal} = E_o S_{BA} E_o^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_{31} & s_{32} \\ s_{41} & s_{42} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$S_{BA-Modal} = \begin{bmatrix} \overbrace{\begin{matrix} (s_{31}+s_{32})+(s_{41}+s_{42}) \\ (s_{31}+s_{32})-(s_{41}+s_{42}) \end{matrix}}^{common-to-common}_{differential-to-common} & \overbrace{\begin{matrix} (s_{31}-s_{32})+(s_{41}-s_{42}) \\ (s_{31}-s_{32})-(s_{41}-s_{42}) \end{matrix}}^{common-to-differential}_{differential-to-differential} \end{bmatrix}$$