MOR via Vector Fitting

- Rational function approximation:
  \[ f(s) \approx \sum_{n=1}^{N} \frac{c_n}{s-a_n} + d + sh \]

- Introduce an unknown function \( \sigma(s) \) that satisfies:
  \[
  \left[ \begin{array}{c}
  \sigma(s)f(s) \\
  \sigma(s)
  \end{array} \right] \approx \left[ \begin{array}{c}
  \sum_{n=1}^{N} \frac{c_n}{s-\tilde{a}_n} + d + sh \\
  \sum_{n=1}^{N} \frac{\tilde{c}_n}{s-\tilde{a}_n} + 1
  \end{array} \right]
  \]

- Poles of \( f(s) \)
  = zeros of \( \sigma(s) \):

- Flip unstable poles into the left half plane.

\[ f(s) \approx \frac{\sum_{n=1}^{N} \frac{c_n}{s-\tilde{a}_n} + d + sh}{\sum_{n=1}^{N} \frac{\tilde{c}_n}{s-\tilde{a}_n} + 1} = \prod_{n=1}^{N} \frac{(s-z_n)}{\prod_{n=1}^{N} (s-\tilde{z}_n)} \]
Blackbox Formulation

Transfer function is approximated

\[ H(\omega) = d + \sum_{k=1}^{L} \frac{c_k}{1 + j\omega / \omega_{ck}} \]

Using curve fitting technique (e.g. vector fitting)

In the time domain, recursive convolution is used

\[ y(t) = dx(t - T) + \sum_{k=1}^{L} y_{pk}(t) \]

where

\[ y_{pk}(t) = a_k x(t - T) \left(1 - e^{-\omega_{ck}T}\right) + e^{-\omega_{ck}T} y_{pk}(t - T) \]

Recursive convolution is fast
Passivity Enforcement

- State-space form: \( \dot{x} = Ax + Bu \)
  \( y =Cx + Du \)

- Hamiltonian matrix:
  \[ M = \begin{bmatrix}
  A + BK^T C & BKB^T \\
  -C^T LC & -A^T - C^T DKB^T
\end{bmatrix} \]

  \[ K = (I - D^T D)^{-1} \]
  \[ L = (I - DD^T)^{-1} \]

- Passive if \( M \) has no imaginary eigenvalues.

- Sweep: \( \text{eig}(I - S(j\omega)^H S(j\omega)) \)

- Quadratic programming:
  - Minimize \( \text{change in response} \) subject to \( \text{passivity compensation} \).

  \[
  \min(\text{vec}(\Delta C)^T H \text{vec}(\Delta C)) \quad \text{subject to} \quad \Delta \lambda = G \cdot \text{vec}(\Delta C). \]
Use of Macromodel

- Time-Domain simulation using recursive convolution
- Frequency-domain circuit synthesis for SPICE netlist
Objective: Determine equivalent circuit from macromodel representation

Motivation

• Circuit can be used in SPICE

Goal

• Generate a netlist of circuit elements
Circuit Realization

Circuit realization consists of interfacing the reduced model with a general circuit simulator such as SPICE

Model order reduction gives a transfer function that can be presented in matrix form as

\[
S(s) = \begin{bmatrix}
s_{11}(s) & \cdots & s_{1N}(s) \\
\vdots & \ddots & \vdots \\
s_{N1}(s) & \cdots & s_{NN}(s)
\end{bmatrix}
\]

or

\[
Y(s) = \begin{bmatrix}
y_{11}(s) & \cdots & y_{1N}(s) \\
\vdots & \ddots & \vdots \\
y_{N1}(s) & \cdots & y_{NN}(s)
\end{bmatrix}
\]
Method 1: Y-Parameter/MOR*

Each of the Y-parameters can be represented as

\[ y_{ij}(s) = d + \sum_{k=1}^{L} \frac{a_k}{s - p_k} \]

where the \( a_k \)'s are the residues and the \( p_k \)'s are the poles. \( d \) is a constant

Equivalent-Circuit Extraction

Macromodel is curve-fit to take the form

\[ Y(s) = d + \sum_{k=1}^{L} \frac{r_k}{s - p_k} \]

Need to find equivalent circuit associated with

- Constant term \(d\)
- Real Poles
- Complex Poles
Y-Parameter - Circuit Realization

Each of the $I_{ij}$ can be realized with a circuit having the following topology:

The resulting current sources can then be superposed for the total current $I_i$ leaving port $i$

*All Y parameters are treated as if they were one-port Y parameters*
Y-Parameter - Circuit Realization

We try to find the circuit associated with each term:

\[ y_{ij}(s) = d_{ij} + \sum_{k=1}^{L} \frac{a_{ijk}}{s - p_{ijk}} \]

For a given port \( i \), the total current due to all ports with voltages \( V_j \) (\( j=1,\ldots,P \)) is given by

\[ I_i = \sum_{j=1}^{P} y_{ij}V_j = \sum_{j=1}^{P} d_{ij}V_j + \sum_{j=1}^{P} V_j \sum_{k=1}^{L} \frac{a_{ijk}}{s - p_{ijk}} \]

For each contributing port, with voltage \( i \), the total current due to a voltage \( V_j \) at port \( j \) is given by

\[ I_{ij} = d_{ij}V_j + V_j \sum_{k=1}^{L} \frac{a_{ijk}}{s - p_{ijk}} \]
Y-Parameter - Circuit Realization

We try to find the circuit associated with each term:

\[ y_{ij}(s) = d + \sum_{k=1}^{L} \frac{a_k}{s - p_k} \]

1. Constant term \(d\)

\[ y_{ijd}(s) = d \]

2. Each pole-residue pair

\[ y_{ijk}(s) = \frac{a_k}{s - p_k} \]
Circuit Realization – Constant Term

\[ R = \frac{1}{d} \]
Circuit Realization – Pole/Residue

In the pole-residue case, we must distinguish two cases

(a) Pole is real
\[ y_{ijk}(s) = \frac{a_k}{s - p_k} \]

(b) Complex conjugate pair of poles
\[ y_{ijk}(s) = \frac{\alpha_k + j\beta_k}{s - \sigma_k - j\omega_k} + \frac{\alpha_k - j\beta_k}{s - \sigma_k + j\omega_k} \]

In all cases, we must find an equivalent circuit consisting of lumped elements that will exhibit the same behavior.
Consider the circuit shown above. The input impedance $Z$ as a function of the complex frequency $s$ can be expressed as:

$$Z = sL + R$$

$Y(s) = \frac{1/L}{s + R/L}$

Comparing $Y(s)$ and $\hat{Y}(s)$ yields the solution

$$L = \frac{1}{a_k}$$

$$R = -\frac{p_k}{a_k}$$
Circuit Realization - Complex Poles

Each term associated with a complex pole pair in the expansion gives:

\[ \hat{Y} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} \]

Where \( r_1, r_2, p_1 \) and \( p_2 \) are the complex residues and poles. They satisfy: \( r_1 = r_2^* \) and \( p_1 = p_2^* \)

It can be re-arranged as:

\[ \hat{Y} = (r_1 + r_2) \left[ \frac{s - (r_1 p_2 + r_2 p_1) / (r_1 + r_2)}{s^2 - s(p_1 + p_2) + p_1 p_2} \right] \]
Circuit Realization - Complex Poles

The pole/residue representation of the Y parameters is given by:

\[
\hat{Y} = (r_1 + r_2) \left[ \frac{s - (r_1 p_2 + r_2 p_1)}{(r_1 + r_2)} \right]
\]

\[
= \frac{s^2 - s(p_1 + p_2) + p_1 p_2}{s^2 - s g + p}
\]

**DEFINE**

\[ p = p_1 p_2 \quad \text{product of poles} \]

\[ a = r_1 + r_2 \quad \text{sum of residues} \]

\[ g = p_1 + p_2 \quad \text{sum of poles} \]

\[ x = r_1 p_2 + r_2 p_1 \quad \text{cross product} \]
Consider the circuit shown above. The input impedance $Z$ as a function of the complex frequency $s$ can be expressed as:

$$Z = sL + R_1 + \frac{1}{1/R_2 + sC} = sL + R_1 + \frac{R_2}{1 + sCR_2}$$

$$Z = \frac{(R_1 + sL)(1 + sCR_2) + R_2}{1 + sCR_2}$$
Circuit Realization - Complex Poles

\[
Y = \frac{CR_2 \left( s + \frac{1}{CR_2} \right)}{LCR_2 \left[ s^2 + s \left( \frac{L + CR_1 R_2}{LR_2 C} \right) + \left( \frac{R_1 + R_2}{LR_2 C} \right) \right]}
\]

\[
Y = \frac{1}{L} \left[ s^2 + s \left( \frac{L + CR_1 R_2}{LR_2 C} \right) + \left( \frac{R_1 + R_2}{LR_2 C} \right) \right]^{-1}
\]
Circuit Realization - Complex Poles

Comparing

\[
Y = \frac{1}{L} \left[ \frac{s + 1/CR_2}{s^2 + s \left( \frac{L + CR_1R_2}{LR_2C} \right) + \left( \frac{R_1 + R_2}{LR_2C} \right)} \right]
\]

with

\[
\hat{Y} = a \left[ \frac{s - x/a}{s^2 - sg + p} \right]
\]

We can identify the circuit elements

\[
L = \frac{1}{a}
\]

\[
R_1 = \frac{x}{a^2} - \frac{g}{a}
\]

\[
R_2 = -\frac{p}{x} - \frac{x}{a^2} + \frac{g}{a}
\]

\[
C = \frac{pa}{x^2} + \frac{1}{a} - \frac{g}{x}
\]
Circuit Realization - Complex Poles

$L = 1 / a$

$R_2 = -\frac{p}{x} - \frac{x}{a^2} + \frac{g}{a}$

$R_1 = \frac{x}{a^2} - \frac{g}{a}$

$C = \frac{pa}{x^2} + \frac{1}{a} - \frac{g}{x}$
Negative Elements

In the circuit synthesis process, it is possible that some circuit elements come as negative. To prevent this situation, we add a contribution to the real parts of the residues of the system. In the case of a complex residue, for instance, assume that

$$\hat{Y} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2}$$

$$\hat{Y} = \frac{r_1 + \Delta}{s-p_1} + \frac{r_2 + \Delta}{s-p_2} - \left( \frac{\Delta}{s-p_1} + \frac{\Delta}{s-p_2} \right)$$

Augmented Circuit

Compensation Circuit

Can show that both augmented and compensation circuits will have positive elements.
Why S Parameters?

Y-Parameter

\[ Y_{11} = \frac{1 + e^{-2\gamma l}}{Z_c (1 - e^{-2\gamma l})} \]

- \( Z_c \): microstrip characteristic impedance
- \( \gamma \): complex propagation constant
- \( l \): length of microstrip

Y\textsubscript{11} can be unstable

S-Parameter

\[ S_{11} = \frac{(1 - e^{-2\gamma l}) \Gamma}{1 - \Gamma^2 e^{-2\gamma l}} \]

\[ \Gamma = \frac{Z_c - Z_o}{Z_c + Z_o} \]

S\textsubscript{11} is always stable
Observation: S-parameters decay rapidly; Y parameters do not.
S = \left( ZZ_o^{-1} + I \right)^{-1} \left( ZZ_o^{-1} - I \right)

Z = [I + S][I - S]^{-1} Z_o

determining

\( Z_0 = \begin{bmatrix} 50.0 & 0.0 & 0.0 & 50.0 \\
0.0 & 50.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 50.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 50.0 \end{bmatrix} \)

as reference...

using

\( Z_0 = \begin{bmatrix} 328.0 & 69.6 & 328.9 & 69.6 \\
69.6 & 328.8 & 69.6 & 328.9 \\
328.9 & 69.6 & 328.8 & 69.6 \\
69.6 & 328.9 & 69.6 & 328.8 \end{bmatrix} \)

as reference...
Method 2: S-Parameter /MOR

\[ A_i(\omega) = \frac{1}{2} \left[ V_i(\omega) + Z_o I_i(\omega) \right] \]

\[ B_i(\omega) = \frac{1}{2} \left[ V_i(\omega) - Z_o I_i(\omega) \right] \]

Need equivalent circuit for \( S_{ijk} \)

All S parameters are treated as if they were one-port S parameters
Strategy

For a given circuit, a relationship between the input admittance $Y_{ijk}(s)$ of the circuit and the associated one-port $S$-parameter representation $S_{ijk}(s)$ can be described by

$$S_{ijk}(s) = \frac{Y_o - Y_{ijk}(s)}{Y_o + Y_{ijk}(s)}$$

$$Y_{ijk}(s) = Y_o \frac{1 - S_{ijk}(s)}{1 + S_{ijk}(s)}$$

$Y_o$ is the reference admittance
Equivalent-Circuit Extraction

Macromodel is curve-fit to take the form

\[ S(s) = d + \sum_{k=1}^{L} \frac{r_k}{s - p_k} \]

Need to find equivalent circuit associated with

- Constant term \( d \)
- Real Poles
- Complex Poles
Constant Term

\[ R = \left( \frac{1 - S_{ijk}}{1 + S_{ijk}} \right) Y_o = \left( \frac{1 - d}{1 + d} \right) Y_o \]

\[ R = Y_o (1 - d) / (1 + d) \]
S- Circuit Realization – Constant Term

\[ R = \frac{Y_o (1-d)}{1+d} \]
S-Parameters - Poles and Residues

In the pole-residue case, we must distinguish two cases

(a) Pole is real

\[ s_{ijk}(s) = \frac{a_k}{s - p_k} \]

(b) Complex conjugate pair of poles

\[ s_{ijk}(s) = \frac{\alpha_k + j\beta_k}{s - \sigma_k - j\omega_k} + \frac{\alpha_k - j\beta_k}{s - \sigma_k + j\omega_k} \]

In all cases, we must find an equivalent circuit consisting of lumped elements that will exhibit the same behavior
S-Realization – Real Poles

Proposed Circuit Model

Admittance of proposed model is given by:

\[ Y = \frac{\left( R_1 + R_2 \right)}{R_1 R_2} \begin{bmatrix} \frac{1}{s + \frac{1}{\left( R_1 + R_2 \right) C}} \\ \frac{1}{R_2 C} \end{bmatrix} \]
Real Poles

\[ Y = \left( \frac{R_1 + R_2}{R_1 R_2} \right) \left[ s + \frac{1}{\left( \frac{R_1 + R_2}{R_2} \right)C} \right] \]

\[ S_{ijk}(s) = \frac{a_k}{s - p_k} \]

\[ \hat{Y} = Y_0 \left( 1 - \frac{S_{ijk}}{1 + S_{ijk}} \right) = Y_o \left( \frac{1 - \frac{r}{s - p}}{1 + \frac{r}{s - p}} \right) = Y_o \left( \frac{s - a}{s + a} \right) \]

from circuit

from pole and residue
Solution for Real Poles

Comparing $\hat{Y}(s)$ with $Y(s)$ gives

$$C = -\frac{(b-a)}{b^2 Z_o}$$

$$R_2 = -\frac{1}{bC}$$

$$R_1 = -R_2 - \frac{1}{aC}$$

where $a = p_k + r_k$, and $b = p_k - r_k$
Realization – Complex Poles

From the S-parameter expansion, the complex pole pair gives:

\[
\hat{S} = \frac{r_1}{s-p_1} - \frac{r_2}{s-p_2} = \frac{s(r_1+r_2)-(r_1p_2+r_2p_1)}{s^2 - s(p_1+p_2) + p_1p_2}
\]

which corresponds to an admittance of:

\[
\hat{Y} = Y_o \left( \frac{1 - \hat{S}}{1 + \hat{S}} \right) = \begin{pmatrix} \frac{sa-x}{s^2 - sg + p} \\ \frac{sa-x}{s^2 - sg + p} \end{pmatrix} Y_o
\]
Realization – Complex Poles

The admittance expression can be re-arranged as

\[
\hat{Y} = \left( \frac{s^2 - sg + p - sa + x}{s^2 - sg + p + sa - x} \right) Y_o = \left( \frac{s^2 - s(g + a) + p + x}{s^2 - s(g - a) + p - x} \right) Y_o
\]

WE HAD DEFINED

\[
p = p_1 p_2 \quad \text{product of poles} \quad a = r_1 + r_2 \quad \text{sum of residues}
\]

\[
g = p_1 + p_2 \quad \text{sum of poles} \quad x = r_1 p_2 + r_2 p_1 \quad \text{cross product}
\]
Realization – Complex Poles

\[ \hat{Y} = \left( \frac{s^2 - sg + p - sa + x}{s^2 - sg + p + sa - x} \right) Y_o = \left( \frac{s^2 - s(g + a) + p + x}{s^2 - s(g - a) + p - x} \right) Y_o \]

This can be further rearranged as

\[ \hat{Y} = \left( \frac{s^2 + sA + B}{s^2 + sD + F} \right) E \]

in which

\[ A = -(g + a), \quad B = p + x, \quad D = -(g - a), \quad F = p - x \]

and

\[ E = Y_o = \frac{1}{H} \]
Realization – Complex Poles

There are several circuit topologies that will work

Model 1

Model 13

Model 9

Model 12
Realization – Complex Poles

More circuit topologies that will work

Model 10

Model 11

Model 8

Model 12
Complex Poles – Model 1

In particular, if we choose model 1

\[ Y = \frac{1}{R_o} \left[ s^2 + s \left( \frac{L + R_1 R_2 C + R_o R_2 C}{LCR_2} \right) + \frac{R_o + R_1 + R_2}{LCR_2} \right] \]

Must be matched with

\[ \hat{Y} = Y_o \left( \frac{1 - S_{ijk}}{1 + S_{ijk}} \right) = \begin{pmatrix} 1 - \frac{sa - x}{s^2 - sg + p} \\ 1 + \frac{sa - x}{s^2 - sg + p} \end{pmatrix} Y_o \]
Complex Poles – Model 1

Matching the terms with like coefficients gives

\[ R_o = \frac{1}{Y_o} \]

\[ p + x = \frac{R_o + R_1 + R_2}{LCR_2} \]

\[ p - x = \frac{R_1 + R_2}{LCR_2} \]

\[ 2p = \frac{R_o + 2R_1 + 2R_2}{LCR_2} \]

\[ 2x = \frac{R_o}{LCR_2} \]
Complex Poles – Model 1

\[ L = -\frac{R_o}{2a} \]

\[ R_2 = \frac{R_o \left( \frac{p}{x} - 1 \right)}{2} - R_1 \]

\[ R_1 = \frac{1}{2} \left( \frac{gR_o}{a} + \frac{2Lx}{a} - R_o \right) \]

\[ C = \frac{-a}{R_2 x} \]
Complex Poles – Model 9

If we choose Model 9

\[
Y_9 = \frac{E}{R_1 + R_3} \frac{A}{R_2 + R_3} \left[ \frac{L + CR_2 R_3 + CR_1 (R_2 + R_3)}{LC(R_1 + R_3)} + \frac{B}{LC(R_1 + R_3)} \right] + \frac{R_2 + R_3}{LC(R_1 + R_3)} \left[ \frac{CR_1 R_2 R_3 + LR_1}{LC R_1 R_3} + \frac{R_1 (R_2 + R_3)}{LC R_1 R_3} \right]
\]

from which the circuit elements can be extracted
Complex Poles – Model 9

\[ R_1 = \frac{F}{BH} \]

\[ R_2 = \frac{\left(-BD^2 + BF + ADF - F^2\right)}{\left(B^2 - ABD + BD^2 + A^2F - 2BF - ADF + F^2\right)H} \]

\[ R_3 = \frac{F}{\left(-B + F\right)H} \]

\[ L = \frac{-BD + AF}{\left(B^2 - ABD + BD^2 + A^2F - 2BF - ADF + F^2\right)H} \]

\[ C = \frac{(-BD + AF)H}{F^2} \]
Special Case – Model 4

A special case exists when $x=0$

or $x = r_1 p_2 + r_2 p_1 = 0$

$$\hat{Y} = \left( \frac{s^2 - s(g + a) + p + x}{s^2 - s(g - a) + p - x} \right) Y_o = \left( \frac{s^2 - s(g + a) + p}{s^2 - s(g - a) + p} \right) Y_o$$

$$\hat{Y} = \left( \frac{s^2 + sA + B}{s^2 + sD + B} \right) E$$

in which

$$A = -(g + a), \quad B = p + x, \quad D = -(g - a), \quad F = p$$

and $$E = Y_o = \frac{1}{H}$$
Special Case – Model 4

The proposed circuit model is

Model 4

This model has one inductor, one capacitor and two resistors.
Special Case – Model 4

The admittance seen at the input is:

\[
Y_4 = \frac{E}{R_1 + R_2} + \frac{1}{R_1 R_2} \left[ \frac{s^2 + s}{C(R_1 + R_2)} + \frac{B=E}{LC} \right]
\]

The elements are extracted as

\[
R_1 = \frac{DE}{A}, \quad R_2 = \frac{R_1 E}{R_1 - E}, \quad C = \frac{1}{D R_2}, \quad L = \frac{1}{F C}
\]
Method 3 - Convolution with S Parameters

In frequency domain  \( B=SA \)

In time domain  \( b(t) = s(t) \ast a(t) \)

Convolution:  \( s(t) \ast a(t) = \int_{-\infty}^{\infty} s(t-\tau)a(\tau)d\tau \)
Most of the computational effort is in the convolution calculation of the history $H(t)$ at each time step.

$$Y_{\text{stamp}} = Z_0^{-1} [1 + s'(0)]^{-1} [1 - s'(0)]$$

$$I_{\text{stamp}} = 2Z_0^{-1} [1 + s'(0)]^{-1} H(t)$$

We make the convolution faster using a $\delta$-function expansion for $s(t)$.
S-Parameter Expansion

In the frequency domain, assume that S parameter can be expanded in the form:

\[ S(l) = \sum_{k=1}^{M} c_k e^{j2\pi lk} \]

In the time domain, this corresponds to:

\[ s(p) = \sum_{k=1}^{M} c_k \delta(p - k) \]

Which is an impulse train of order \( M \). Convolution will then give:

\[ y(p) = s(p) * x(p) = \left[ \sum_{k=1}^{M} c_k \delta(p - k) \right] * x(p) = \sum_{k=1}^{M} c_k x(p - k) \]

If we truncate the summation to the \( Q \) largest impulses:

\[ y(p) = \left[ \sum_{k=1}^{Q} c_k \delta(p - k) \right] * x(p) = \sum_{k=1}^{Q} c_k x(p - k) \]

*If the reference system is optimized*, most of the coefficients \( c_k \) will be negligibly small; therefore \( \Rightarrow \) \( Q \) will be a relatively small number

**Fast Convolution is achieved by making \( Q \) small**
Strategy

\[ s_{ijk}(u) = c_{ijk} \delta(u - k) \]

Since the S-parameter is a simple delay, the one-port circuit representation is a transmission line with characteristic impedance \( Z_0 \) and delay \( k \).
Circuit Interpretation

\[ s_{ijk}(u) = c_{ijk} \delta(u - k) \]

\[ \tau_d = k \Delta t \]

\[ \Delta t = \text{time step} \]
Method 1: Y-Parameters/MOR

Recursive convolution

SPICE realization
Method 2: S-Parameters/MOR

Recursive convolution

SPICE realization
Comparisons

SPICE simulation from MOR generated Netlist (Method 2)

Direct Convolution

**SPICE Circuit**

**Fast Convolution**

![Graph 1](image1.png)

![Graph 2](image2.png)
Coupled Lines
(4-port)

Direct

SPICE simulation
Using generated netlist
(Method 2)
Comparison of S-methods

SPICE Netlist from MOR with S (Method 2)

SPICE Netlist from Fast convolution with S (Method 3)

Convolution (no SPICE netlist)