ECE 546
Lecture -16
MNA and SPICE

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The Node Voltage method consists in determining potential differences between nodes and ground (reference) using KCL.

For Node 1:

\[
\frac{V_1 - V_m}{Z_A} + \frac{V_1}{Z_B} + \frac{V_1 - V_2}{Z_C} = 0
\]
Nodal Analysis

For Node 2:

$$\frac{V_2 - V_1}{Z_C} + \frac{V_2}{Z_D} + \frac{V_2 - V_n}{Z_E} = 0$$

Rearranging the terms gives:

$$\left(\frac{1}{Z_A} + \frac{1}{Z_B} + \frac{1}{Z_C}\right)V_1 - \left(\frac{1}{Z_C}\right)V_2 = \left(\frac{1}{Z_A}\right)V_m$$

$$-\left(\frac{1}{Z_C}\right)V_1 + \left(\frac{1}{Z_C} + \frac{1}{Z_D} + \frac{1}{Z_E}\right)V_2 = -\left(\frac{1}{Z_E}\right)V_n$$

Defining:

$$G_A = \frac{1}{Z_A}, \ G_B = \frac{1}{Z_B}, \ G_C = \frac{1}{Z_C}, \ G_D = \frac{1}{Z_D}, \ G_E = \frac{1}{Z_E}$$
Rearranging the terms gives:

\[
\begin{bmatrix}
(G_A + G_B + G_C) & -G_C \\
-G_C & (G_C + G_D + G_E)
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= \begin{bmatrix}
G_A V_m \\
-G_E V_m
\end{bmatrix}
\]

The system can be solved to yield \( V_1 \) and \( V_2 \).
Nodal Analysis

For Node 1:
\[
\frac{V_1 - 5\angle0^\circ}{10} + \frac{V_1 + j10\angle45^\circ}{j5} + \frac{V_1 - V_2}{2 + j2} = 0
\]

For Node 2:
\[
\frac{V_2 - V_1}{2 + j2} + \frac{V_2}{3 - j4} + \frac{V_2}{5} = 0
\]
Rearranging the terms gives:

\[
\left(\frac{1}{10} + \frac{1}{j5} + \frac{1}{2+j2}\right)V_1 - \left(\frac{1}{2+j2}\right)V_2 = \frac{5 \angle 0^\circ}{10} - \frac{10 \angle 45^\circ}{j5}
\]

\[
-\left(\frac{1}{2+j2}\right)V_1 + \left(\frac{1}{2+j2} + \frac{1}{3-j4} + \frac{1}{5}\right)V_2 = 0
\]
Nodal Analysis

\[
\begin{bmatrix}
\left( \frac{1}{5} + \frac{1}{j2} + \frac{1}{4} \right) & -\left( \frac{1}{4} \right) \\
-\left( \frac{1}{4} \right) & \left( \frac{1}{4} + \frac{1}{-j2} + \frac{1}{2} \right)
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= 
\begin{bmatrix}
50 \angle 0^\circ \\
50 \angle 90^\circ
\end{bmatrix}
\]

\[[Y] \quad [v] = [i] \]
Nodal Analysis - Solution

\[ V_1 = \begin{vmatrix} 10 & 0.25 \\ j25 & 0.75 + j0.5 \\ 0.45 - j0.5 & -0.25 \\ 0.25 & 0.75 + j0.5 \end{vmatrix} = \frac{13.5 \angle 56.3^\circ}{0.546 \angle -15.95^\circ} = 24.7 \angle 72.25^\circ V \]

\[ V_2 = \begin{vmatrix} 0.45 - j0.5 & 10 \\ -0.25 & j25 \\ 0.45 - j0.5 & -0.25 \\ 0.25 & 0.75 + j0.5 \end{vmatrix} = \frac{18.35 \angle 37.8^\circ}{0.546 \angle -15.95^\circ} = 33.6 \angle 53.75^\circ V \]
Nodal Formulation

Node 1: \(-3 + \frac{V_1 - V_2}{5} = 0\)

Node 2: \(\frac{V_2 - V_1}{5} + \frac{V_2}{10} + \frac{V_2 - V_3}{5} = 0\)

Node 3: \(\frac{V_3 - V_2}{5} + \frac{V_3}{10} = 0\)
Nodal Solution

Arrange in matrix form: \([G][V]=[I]\)

\[
\begin{bmatrix}
0.2 & -0.2 & 0 \\
-0.2 & 0.5 & -0.2 \\
0 & -0.2 & 0.3
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= \begin{bmatrix}
3 \\
0 \\
0
\end{bmatrix}
\]

Use Gaussian elimination to form an upper triangular matrix

\[
\begin{bmatrix}
0.2 & -0.2 & 0 \\
0 & 0.3 & -0.2 \\
0 & 0 & 0.25
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= \begin{bmatrix}
3 \\
3 \\
3
\end{bmatrix}
\]

Solve for \(V_1, V_2\) and \(V_3\) using backward substitution

This can always be solved no matter how large the matrix is
Why SPICE?

- Established platform
- Powerful engine
- Source code available for free
- Extensive libraries of devices
- New device installation procedure easy
### MOS SPICE Parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{\text{drawn}}$</td>
<td>Device length (drawn)</td>
<td>0.35</td>
<td>μm</td>
</tr>
<tr>
<td>$L_{\text{eff}}$</td>
<td>Device length (effective)</td>
<td>0.25</td>
<td>μm</td>
</tr>
<tr>
<td>$t_{\text{ox}}$</td>
<td>Gate oxide thickness</td>
<td>70</td>
<td>A</td>
</tr>
<tr>
<td>$N_a$</td>
<td>Density of acceptor ions in NFET channel</td>
<td>$1.0 \times 10^{17}$</td>
<td>cm(^{-3})</td>
</tr>
<tr>
<td>$N_d$</td>
<td>Density of donor ions in PFET channel</td>
<td>$2.5 \times 10^{17}$</td>
<td>cm(^{-3})</td>
</tr>
<tr>
<td>$V_{Tn}$</td>
<td>NFET threshold voltage</td>
<td>0.5</td>
<td>V</td>
</tr>
<tr>
<td>$V_{Tp}$</td>
<td>PFET threshold voltage</td>
<td>-0.5</td>
<td>V</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Channel modulation parameter</td>
<td>0.1</td>
<td>V(^{-1})</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Body effect parameter</td>
<td>0.3</td>
<td>V(^{1/2})</td>
</tr>
<tr>
<td>$V_{\text{sat}}$</td>
<td>Saturation velocity</td>
<td>$1.7 \times 10^5$</td>
<td>m/s</td>
</tr>
<tr>
<td>$\mu_n$</td>
<td>Electron mobility</td>
<td>400</td>
<td>cm(^2)/Vs</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td>Hole mobility</td>
<td>100</td>
<td>cm(^2)/Vs</td>
</tr>
<tr>
<td>$k_n$</td>
<td>NFET process transconductance</td>
<td>200</td>
<td>μA/V(^2)</td>
</tr>
<tr>
<td>$k_p$</td>
<td>PFET process transconductance</td>
<td>50</td>
<td>μA/V(^2)</td>
</tr>
<tr>
<td>$C_{\text{ox}}$</td>
<td>Gate oxide capacitance per unit area</td>
<td>5</td>
<td>fF/μm(^2)</td>
</tr>
<tr>
<td>$C_{\text{GSO}}C_{\text{GDO}}$</td>
<td>Gate source and drain overlap capacitance</td>
<td>0.1</td>
<td>fF/μm</td>
</tr>
<tr>
<td>$C_j$</td>
<td>Junction capacitance</td>
<td>0.5</td>
<td>fF/μm(^2)</td>
</tr>
<tr>
<td>$C_{\text{JSW}}$</td>
<td>Junction sidewall capacitance</td>
<td>0.2</td>
<td>fF/μm</td>
</tr>
<tr>
<td>$R_{\text{poly}}$</td>
<td>Gate sheet resistance</td>
<td>4</td>
<td>Ω/square</td>
</tr>
<tr>
<td>$R_{\text{diff}}$</td>
<td>Source and drain sheet resistance</td>
<td>4</td>
<td>Ω/square</td>
</tr>
</tbody>
</table>
Problems

• **Nonlinear Devices**
  - Diodes, transistors cannot be simulated in the frequency domain
  - Capacitors and inductors are best described in the frequency domain
  - Use time-domain representation for reactive elements (capacitors and inductors)

• **Circuit Size**
  - Matrix size becomes prohibitively large
Motivations

- Loads are nonlinear
- Need to model reactive elements in the time domain
- Generalize to nonlinear reactive elements
Time-Domain Model for Linear Capacitor

For linear capacitor $C$ with voltage $v$ and current $i$ which must satisfy

$$i = C \frac{dv}{dt}$$

Using the backward Euler scheme, we discretize time and voltage variables and obtain at time $t = nh$

$$v_{n+1} = v_n + hv'_{n+1}$$
Time-Domain Model for Linear Capacitor

After substitution, we obtain

\[ v'_{n+1} = \frac{i_{n+1}}{C} \]

so that

\[ v_{n+1} = v_n + h \frac{i_{n+1}}{C} \]

The solution for the current at \( t_{n+1} \) is, therefore,

\[ i_{n+1} = \frac{C}{h} v_{n+1} - \frac{C}{h} v_n \]
Time-Domain Model for Linear Capacitor

Backward Euler companion model at $t=nh$

Trapezoidal companion model at $t=nh$
Time-Domain Model for Linear Capacitor

Step response comparisons
**Time-Domain Model for Linear Inductor**

\[ v = L \frac{di}{dt} \]

**Backward Euler:**

\[ i_{n+1} = i_n + hi'_{n+1} \]

\[ i'_{n+1} = \frac{v_{n+1}}{L} \]

\[ v_{n+1} = \frac{L}{h}i_{n+1} - \frac{L}{h}i_n \]
Time-Domain Model for Linear Inductor

If trapezoidal method is applied

\[ i_{n+1} = i_n + \frac{h}{2} \left[ i_{n+1}' + i_n' \right] \]

\[ v_{n+1} = \frac{2L}{h} i_{n+1} - \left( \frac{2L}{h} + v_n \right) \]
The Diode

- **Diode Properties**
  - Two-terminal device that conducts current freely in one direction but blocks current flow in the opposite direction.
  - The two electrodes are the anode which must be connected to a positive voltage with respect to the other terminal, the cathode in order for current to flow.
Diode Circuits

\[ V_{out} = V_D \]
\[ I_D = I_S \left( e^{V_D / V_T} - 1 \right) \]
\[ V_S = R I_D + V_D = R I_D (V_D) + V_D \]

Nonlinear transcendental system ➔ Use graphical method

Solution is found at intersection of load line characteristics and diode characteristics
BJT Ebers-Moll Model

NPN Transistor

\[ i_E = \left( \frac{I_S}{\alpha_F} \right) \left( e^{v_{BE}/V_T} - 1 \right) - I_S \left( e^{v_{BC}/V_T} - 1 \right) \]

\[ i_C = I_S \left( e^{v_{BE}/V_T} - 1 \right) - \left( \frac{I_S}{\alpha_R} \right) \left( e^{v_{BC}/V_T} - 1 \right) \]

\[ i_B = \left( \frac{I_S}{\beta_F} \right) \left( e^{v_{BE}/V_T} - 1 \right) + \left( \frac{I_S}{\beta_R} \right) \left( e^{v_{BC}/V_T} - 1 \right) \]

\[ \beta_F = \frac{\alpha_F}{1 - \alpha_F} \quad \beta_R = \frac{\alpha_R}{1 - \alpha_R} \]

Describes BJT operation in all of its possible modes
Problem: Wish to solve for $f(x) = 0$

Use fixed point iteration method:

Define $F(x) = x - K(x)f(x)$

$\exists: \quad x_{k+1} = F(x_k) = x_k - K(x_k)f(x_k)$

With Newton Raphson: $K(x) = [f'(x)]^{-1} = \left[\frac{df}{dx}\right]^{-1}$

therefore, $\exists: \quad x_{k+1} = x_k - [f'(x_k)]^{-1}f(x_k)$
Newton Raphson Method

(Graphical Interpretation)
Newton Raphson Algorithm

\[ N - R : \quad x_{k+1} = x_k - A_k^{-1} f(x_k) \]

\[ A_k x_{k+1} = A_k x_k - f(x_k) = S_k. \]

\( x_{k+1} \) is the solution of a linear system of equations.

\[ A_k x = S_k \quad \leftarrow \quad \text{LU fact} \]

Forward and backward substitution.

\( A_k \) is the nodal matrix for \( N_k \)

\( S_k \) is the rhs source vector for \( N_k \).
Newton Raphson Algorithm

0. \( k \rightarrow 0 \), gives \( \vec{V}_0 \), \( \vec{i}_0 \)

1. Find \( V_k, i_k \) compute companion models.
   \[
   G_k, I_k, R_k, E_k
   \]
   \[
   V_c, C_c
   \]

2. Obtain \( A_k, S_k \).

3. Solve \( A_k x_k = S_k \).

4. \( x_{k+1} \leftarrow \text{Solution} \)

5. Check for convergence \( \| x_{k+1} - x_k \| < \varepsilon \).
   
   If they converge, then stop.

6. \( k + 1 \rightarrow k \), and go to step 1.
Newton Raphson - Diode

It is obvious from the circuit that the solution must satisfy
\[ f(V) = 0 \]
We also have
\[ f'(V) = \frac{1}{R} + \frac{I_s}{V_t} e^{V/V_t} \]

The Newton method relates the solution at the (k+1)th step to the solution at the kth step by

\[ V_{k+1} = V_k - \frac{f(V_k)}{f'(V_k)} + V_k \]

\[ V_{k+1} = V_k - \frac{V_k - E}{R} + I_s \left( e^{V_k/V_t} - 1 \right) \]

\[ V_{k+1} = V_k - \frac{1}{R} + \frac{I_s}{V_t} e^{V_k/V_t} \]
After manipulation we obtain

\[
\left( \frac{1}{R} + g_k \right) V_{k+1} = \frac{E}{R} - J_k
\]

\[
g_k = \frac{I_s}{V_t} \frac{e^{V_k/V_t}}{e^{V_k/V_t} - 1} - V_k g_k
\]
Diode Circuit – Iterative Method

Newton-Raphson Method

Use: \[ x_{k+1} = x_k - \left[ f'(x_k) \right]^{-1} f(x_k) \]

\[ x^{(k+1)} = x^{(k)} - \left[ f'(x^{(k)}) \right]^{-1} f(x^{(k)}) \]

\[ f(V_D) = \frac{V_D - V_S}{R} + I_S \left( e^{V_D/V_T} - 1 \right) = 0 \]

\[ f'(V_D) = \frac{1}{R} + \frac{I_S}{V_T} e^{V_D/V_T} \]

\[ V_D^{(k+1)} = V_D^{(k)} - \frac{V_D^{(k)} - V_S}{R} + I_S \left( e^{V_D^{(k)}/V_T} - 1 \right) \]

Where \( V_D^{(k)} \) is the value of \( V_D \) at the \( k \)th iteration

Procedure is repeated until convergence to final (true) value of \( V_D \) which is the solution. Rate of convergence is quadratic.
Newton Raphson for Diode

Newton-Raphson representation of diode circuit at kth iteration

\[
g_k = \frac{I_s}{V_t} e^{V_{k}/V_t}
\]

\[
J_k = I_s (e^{V_{k}/V_t} - 1) - V_k g_k
\]
Current Controlled

\[ R_k = \frac{dh(i)}{di} \bigg|_{i=i_k} \]

\[ E_k = h(i_k) - R_k i_k \]
General Network

Let $x = \text{vector variables in the network to be solved for}$. Let $f(x) = 0$ be the network equations. Let $x_k$ be the present iterate, and define

$$A_k = f'(x_k) \rightarrow \text{Jacobian of } f \text{ at } x = x_k$$

Let $N_k$ be the linear network where each non-linear resistor is replaced by its companion model computed from $x_k$.

$$I_j = g_j(V_j)$$
General Network

\[ V_{jk} = P_{j+k} - P_{j-k} \]

Companion model

\[ G_k = \left. \frac{d g(V)}{dV} \right|_{V=V_k} \]

\[ I_k = g[V_k] - G_k V_k \]
Nonlinear Reactive Elements

\[ q = f(v), \quad i = \frac{dq}{dt} \]

\[ q_{n+1} = q_n + h \left. \frac{dq}{dt} \right|_{t=t_{n+1}} \]

or, \[ i_{n+1} = \frac{q_{n+1}}{h} - \frac{q_n}{h} \Rightarrow i_{n+1}(v_{n+1}) = \frac{f(v_{n+1})}{h} \]
General Element

\[ I = f(V) \]

slope = \( g_k \)

\[ I_k \]

\[ V_k \]

\[ J_k \]

\[ i_{k+1} \]

\[ v_{k+1} \]

\[ g_k \]

\[ J_k \]