Transmission Lines

1 Single transmission-line equations

Consider the basic telegrapher's equations for a single ideal (lossless, dispersionless) transmission line as a function of the time \( t \) and the position \( x \) along the line (see Figure 1).

\[
-\frac{\partial V}{\partial x} = L \frac{\partial I}{\partial t} \tag{1a}
\]

\[
-\frac{\partial I}{\partial x} = C \frac{\partial V}{\partial t} \tag{1b}
\]

where \( V \) and \( I \) are the voltage and current, respectively, and \( L \) and \( C \) are the inductance and capacitance per unit length. Solutions must be sought for the voltage and current which

![Figure 1. (a) Basic electrical model for lossless transmission line. (b) Representation of a single ideal lossless microstrip line with linear resistive terminations.](image-url)
satisfy the boundary conditions at \( x=0 \) and at \( x=l \), where \( l \) is the length of the line. First, Equations (1) are combined to yield the second-order differential equations

\[
\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}
\]  

(2a)

\[
\frac{\partial^2 I}{\partial x^2} = CL \frac{\partial^2 I}{\partial t^2}
\]  

(2b)

For matters of convenience, we need to find a frequency-domain solution for a time-harmonic excitation of frequency \( f = \omega/2\pi \) where \( \omega \) is the angular frequency. This implies that \( V \) and \( I \) have \( e^{j\omega t} \) time dependence; therefore, Equations (2) can be written as

\[
\frac{\partial^2 V}{\partial x^2} = -\omega^2 LC V
\]  

(3a)

\[
\frac{\partial^2 I}{\partial x^2} = -\omega^2 CLI
\]  

(3b)

The general solutions for the voltage and the current are given by

\[
V(x) = Ae^{-j\omega x/vo} + Be^{j\omega x/vo}
\]  

(4a)

\[
I(x) = \frac{A}{Zo} e^{-j\omega x/vo} - \frac{B}{Zo} e^{j\omega x/vo}
\]  

(4b)

where the characteristic impedance \( Zo \) is given by

\[
Zo = \sqrt{\frac{L}{C}}
\]  

(5)

and the propagation velocity \( vo \) is given by

\[
vo = \frac{l}{\sqrt{LC}}.
\]  

(6)
Equations (4) are superpositions of forward and backward wave solutions to Equations (3) and the coefficients A and B are obtained by matching the boundary conditions at x=0 and x=l (see Figure 1(b)), namely,

\[ V_1(\omega) = V(0) + Z_1 I(0) \]  \hspace{1cm} (7a)

\[ 0 = V(l) - Z_2 I(l) . \]  \hspace{1cm} (7b)

\( V_1(\omega) \) is the source voltage provided at x=0. By substituting the above conditions in Equations (4), we get the values for the coefficients A and B.

\[ A = T \frac{V_1(\omega)}{1 - \Gamma_1 \Gamma_2 e^{-2j\omega l/v_0}} \]  \hspace{1cm} (8a)

\[ B = \Gamma_2 e^{-2j\omega l/v_0} A \]  \hspace{1cm} (8b)

where \( T_1 \) is the transmission coefficient; \( \Gamma_1 \) and \( \Gamma_2 \) are the source and load reflection coefficients respectively; they are given by

\[ T = \frac{Z_o}{Z_1 + Z_o} \]  \hspace{1cm} (9)

\[ \Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o} \]  \hspace{1cm} (10)

\[ \Gamma_2 = \frac{Z_2 - Z_o}{Z_2 + Z_o} . \]  \hspace{1cm} (11)

A and B can be substituted into Equations (4) in order to get the complete solutions. It is of interest to observe that the magnitudes of \( \Gamma_1 \) and \( \Gamma_2 \) are less than 1; therefore, the denominator of Equation (8a) can be expanded into a geometric series which yields

\[ A = TV_1(\omega) \sum_{k=0}^{\infty} \Gamma_1^k \Gamma_2^k e^{-2j\omega k l/v_0} \]  \hspace{1cm} (12a)
\[ B = TV_1(\omega) \sum_{k=0}^{\infty} \Gamma_1^k \Gamma_2^{k+1} e^{-j\omega(k+1)v_o} \]  

(12b)

Therefore, the frequency-domain solutions are described by forward and backward wave solutions given respectively by

\[
V_f(x) = \sum_{k=0}^{\infty} \Gamma_1^k \Gamma_2^{k+1} e^{-2j\omega k l / v_o} e^{-j\omega x / v_o} TV_1(\omega)
\]  

(13a)

\[
V_b(x) = \sum_{k=0}^{\infty} \Gamma_1^k \Gamma_2^{k+1} e^{-2j\omega(k+1) l / v_o} e^{+j\omega x / v_o} TV_1(\omega)
\]  

(13b)

and the total voltage is given by

\[
V(x) = V_f(x) + V_b(x).
\]  

(14)

Several important features need to be stressed regarding Equations (13). First, we notice that for moderate reflection coefficients, only a few terms of the infinite series need to be retained in order to obtain an accurate approximation. Second, by recognizing that \( V_1(\omega) \) can be regarded as the Fourier transform of some arbitrary time-domain voltage, \( V_1(t) \), and by recalling that \( e^{-j\omega t} \) is the Fourier transform of the delayed impulse function \( \delta(t-\tau) \), inversion of Equation (13) can easily be performed into the time domain:

\[
V_f(x,t) = T \sum_{k=0}^{\infty} \Gamma_1^k \Gamma_2^{k+1} V_1 \left( t - \frac{x + 2k l}{v_o} \right)
\]  

(15a)

\[
V_f(x,t) = T \sum_{k=0}^{\infty} \Gamma_1^k \Gamma_2^{k+1} V_1 \left( t - \frac{x - 2(k+1) l}{v_o} \right)
\]  

(15b)

\[
V(x,t) = V_f(x,t) + V_b(x,t).
\]  

(16)
Equations (15) and (16) represent the general time-domain solution for an arbitrary excitation, $V_1(t)$. As can be seen, they are made of a superposition of delayed and attenuated components of the original signal $V_1(t)$.