ECE 451
Circuit Synthesis

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MOR via Vector Fitting

- Rational function approximation:

\[ f(s) \approx \sum_{n=1}^{N} \frac{c_n}{s-a_n} + d + sh \]

- Introduce an unknown function \( \sigma(s) \) that satisfies:

\[
\left[ \sigma(s)f(s) \right] = \left[ \sum_{n=1}^{N} \frac{c_n}{s-a_n} + d + sh \right]
\]

\[
\sigma(s) = \sum_{n=1}^{N} \left( \tilde{c}_n + 1 \right)
\]

- Poles of \( f(s) \) = zeros of \( \sigma(s) \):

- Flip unstable poles into the left half plane.
Passivity Enforcement

- State-space form: \[ \dot{x} = Ax + Bu \]
  \[ y = Cx + Du \]

- Hamiltonian matrix:
  \[ M = \begin{bmatrix} A + BKD^T C & BKB^T \\ -C^T LC & -A^T - C^T DKB^T \end{bmatrix} \]
  \[ K = (I - DD^T)^{-1} \]
  \[ L = (I - DD^T)^{-1} \]

- Passive if \( M \) has no imaginary eigenvalues.

- Sweep: \( \text{eig} \left( I - S(j\omega)H S(j\omega) \right) \)

- Quadratic programming:
  - Minimize (change in response) subject to (passivity compensation).

\[ \text{min} \left( \text{vec}(\Delta C)^T H \text{vec}(\Delta C) \right) \quad \text{subject to} \quad \Delta \lambda = G \cdot \text{vec}(\Delta C). \]
Macromodel Circuit Synthesis

Use of Macromodel

- Time-Domain simulation using recursive convolution
- Frequency-domain circuit synthesis for SPICE netlist
Macromodel Circuit Synthesis

Objective: Determine equivalent circuit from macromodel representation*

Motivation

• Circuit can be used in SPICE

Goal

• Generate a netlist of circuit elements

Circuit Realization

Circuit realization consists of interfacing the reduced model with a general circuit simulator such as SPICE.

Model order reduction gives a transfer function that can be presented in matrix form as

\[
S(s) = \begin{bmatrix}
s_{11}(s) & \cdots & s_{1N}(s) \\
\vdots & \ddots & \vdots \\
s_{N1}(s) & \cdots & s_{NN}(s)
\end{bmatrix}
\]

or

\[
Y(s) = \begin{bmatrix}
y_{11}(s) & \cdots & y_{1N}(s) \\
\vdots & \ddots & \vdots \\
y_{N1}(s) & \cdots & y_{NN}(s)
\end{bmatrix}
\]
Y-Parameter - Circuit Realization

Each of the Y-parameters can be represented as

\[ y_{ij}(s) = d + \sum_{k=1}^{L} \frac{a_k}{s - p_k} \]

where the \(a_k\)'s are the residues and the \(p_k\)'s are the poles. \(d\) is a constant.
Y-Parameter - Circuit Realization

The realized circuit will have the following topology:

We need to determine the circuit elements within $y_{ijk}$.
Y-Parameter - Circuit Realization

We try to find the circuit associated with each term:

\[ y_{ij}(s) = d + \sum_{k=1}^{L} \frac{a_k}{s - p_k} \]

1. Constant term \(d\)

\[ y_{ijd}(s) = d \]

2. Each pole-residue pair

\[ y_{ijk}(s) = \frac{a_k}{s - p_k} \]
Y-Parameter - Circuit Realization

In the pole-residue case, we must distinguish two cases

(a) Pole is real

\[ y_{ijk}(s) = \frac{a_k}{s - p_k} \]

(b) Complex conjugate pair of poles

\[ y_{ijk}(s) = \frac{\alpha_k + j\beta_k}{s - \sigma_k - j\omega_k} + \frac{\alpha_k - j\beta_k}{s - \sigma_k + j\omega_k} \]

In all cases, we must find an equivalent circuit consisting of lumped elements that will exhibit the same behavior.
Circuit Realization – Constant Term

\[ R = \frac{1}{d} \]
Consider the circuit shown above. The input impedance $Z$ as a function of the complex frequency $s$ can be expressed as:

$$Z = sL + R$$

$$Y(s) = \frac{1}{L} \frac{1}{s + R/L}$$

$$y_{ijk}(s) = \frac{a_k}{s - p_k}$$

$L = 1 / a_k$

$R = -p_k / a_k$
Consider the circuit shown above. The input impedance $Z$ as a function of the complex frequency $s$ can be expressed as:

$$Z = sL + R_1 + \frac{1}{\frac{1}{R_2} + sC} = sL + R_1 + \frac{R_2}{1 + sCR_2}$$

$$Z = \frac{(R_1 + sL)(1 + sCR_2) + R_2}{1 + sCR_2}$$
Circuit Realization - Complex Poles

\[ Y = \frac{CR_2 \left( s + \frac{1}{CR_2} \right)}{LCR_2 \left[ s^2 + s \left( \frac{L + CR_1 R_2}{LR_2 C} \right) + \frac{R_1 + R_2}{LR_2 C} \right]} \]

\[ Y = \frac{1}{L} \frac{\left( s + \frac{1}{CR_2} \right)}{s^2 + s \left( \frac{L + CR_1 R_2}{LR_2 C} \right) + \frac{R_1 + R_2}{LR_2 C}} \]
Circuit Realization - Complex Poles

Each term associated with a complex pole pair in the expansion gives:

\[
\hat{Y} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2}
\]

Where \( r_1, r_2, p_1 \) and \( p_2 \) are the complex residues and poles. They satisfy: \( r_1 = r_2^* \) and \( p_1 = p_2^* \)

It can be re-arranged as:

\[
\hat{Y} = (r_1 + r_2) \left[ \frac{s - (r_1 p_2 + r_2 p_1) / (r_1 + r_2)}{s^2 - s(p_1 + p_2) + p_1 p_2} \right]
\]
Circuit Realization - Complex Poles

We next compare

\[ Y = \frac{1}{L} \left( \frac{s + 1/CR_2}{s^2 + s \left( \frac{L + CR_1 R_2}{LR_2 C} \right) + \frac{R_1 + R_2}{LR_2 C}} \right) \]

and

\[ \hat{Y} = (r_1 + r_2) \left[ \frac{s - (r_1 p_2 + r_2 p_1)/(r_1 + r_2)}{s^2 - s(p_1 + p_2) + p_1 p_2} \right] \]

**DEFINE**

\[ p = p_1 p_2 \quad \text{product of poles} \quad a = r_1 + r_2 \quad \text{sum of residues} \]

\[ g = p_1 + p_2 \quad \text{sum of poles} \quad x = r_1 p_2 + r_2 p_1 \quad \text{cross product} \]
Circuit Realization - Complex Poles

We can identify the circuit elements

\[ L = \frac{1}{a} \]

\[ R_1 = \frac{x}{a^2} - \frac{g}{a} \]

\[ R_2 = -\frac{p}{x} - \frac{x}{a^2} + \frac{g}{a} \]

\[ C = \frac{pa}{x^2} + \frac{1}{a} - \frac{g}{x} \]
Circuit Realization - Complex Poles

In the circuit synthesis process, it is possible that some circuit elements come as negative. To prevent this situation, we add a contribution to the real parts of the residues of the system. In the case of a complex residue, for instance, assume that

\[
\hat{Y} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2}
\]

\[
\hat{Y} = \frac{r_1 + \Delta}{s-p_1} + \frac{r_2 + \Delta}{s-p_2} - \left(\frac{\Delta}{s-p_1} + \frac{\Delta}{s-p_2}\right)
\]

Augmented Circuit

Compensation Circuit

Can show that both augmented and compensation circuits will have positive elements
S-Parameter - Circuit Realization

Each of the S-parameters can be represented as

\[ s_{ij}(s) = d + \sum_{k=1}^{L} \frac{a_k}{s - p_k} \]

where the \( a_k \)'s are the residues and the \( p_k \)'s are the poles. \( d \) is a constant.
Realization from S-Parameters

The realized circuit will have the following topology:

We need to determine the circuit elements within $s_{ijk}$
S-Parameter - Circuit Realization

We try to find the circuit associated with each term:

\[ s_{ij}(s) = d + \sum_{k=1}^{L} \frac{a_k}{s - p_k} \]

1. Constant term \( d \)

\[ s_{ijd}(s) = d \]

2. Each pole and residue pair

\[ s_{ijk}(s) = \frac{a_k}{s - p_k} \]
S-Parameter - Circuit Realization

In the pole-residue case, we must distinguish two cases

(a) Pole is real

\[ s_{ijk}(s) = \frac{a_k}{s - p_k} \]

(b) Complex conjugate pair of poles

\[ s_{ijk}(s) = \frac{\alpha_k + j\beta_k}{s - \sigma_k - j\omega_k} + \frac{\alpha_k - j\beta_k}{s - \sigma_k + j\omega_k} \]

In all cases, we must find an equivalent circuit consisting of lumped elements that will exhibit the same behavior.
S- Circuit Realization – Constant Term

\[ R = Y_o \left( \frac{1 - d}{1 + d} \right) \]
S-Realization – Real Poles

\[ S_{ijd} \]

\[ R_1 \quad R_2 \quad C \]
S-Realization – Real Poles

Admittance of proposed model is given by:

\[ Y = \frac{R_1 + R_2}{R_1 \cdot R_2} \left[ s + \frac{1}{\left( R_1 + R_2 \right) C} \right] \]
S-Realization – Real Poles

From S-parameter expansion we have:

\[ s_{ijk}(s) = \frac{r_k}{s - p_k} \]

which corresponds to:

\[ \hat{Y} = Y_o \left( \frac{s - a}{s - b} \right) \text{ where } a = p_k + r_k, \text{ and } b = p_k - \]

from which

\[ Y_o = \frac{R_1 + R_2}{R_1 R_2} \]

\[ C = -\frac{(b - a)}{b^2 Z_o} \]

\[ R_2 = \frac{-1}{bC} \]

\[ R_1 = -R_2 - \frac{1}{aC} \]
Realization – Complex Poles

Proposed model

\[ Y = Y_a + \frac{1}{R_o} = \frac{1 + sCR_2}{s^2LCR_2 + s(L + R_1R_2C) + (R_1 + R_2)} + \frac{1}{R_o} \]

which can be re-arranged as:

\[ Y = \frac{1}{R_o} \left[ \frac{s^2 + s \left( \frac{L + R_1R_2C + R_oR_2C}{LCR_2} \right)}{s^2 + s \left( \frac{L + R_1R_2C}{LCR_2} \right)} + \frac{R_o + R_1 + R_2}{LCR_2} \right] \]
Realization – Complex Poles

From the S-parameter expansion, the complex pole pair gives:

\[
\hat{S} = \frac{r_1}{s - p_1} - \frac{r_2}{s - p_2} = \frac{s(r_1 + r_2) - (r_1 p_2 + r_2 p_1)}{s^2 - s(p_1 + p_2) + p_1 p_2}
\]

which corresponds to an admittance of:

\[
\hat{Y} = Y_o \left( \frac{1 - \hat{S}}{1 + \hat{S}} \right) = \begin{pmatrix}
1 - \frac{sa - x}{s^2 - sg + p} \\
\frac{s^2 - sg + p}{sa - x} \\
1 + \frac{sa - x}{s^2 - sg + p}
\end{pmatrix} Y_o
\]
Realization – Complex Poles

The admittance expression can be re-arranged as

$$\hat{Y} = \left( \frac{s^2 - sg + p - sa + x}{s^2 - sg + p + sa - x} \right) Y_o = \left( \frac{s^2 - s(g + a) + p + x}{s^2 - s(g - a) + p - x} \right) Y_o$$

**WE HAD DEFINED**

$$p = p_1 p_2 \quad \text{product of poles} \quad a = r_1 + r_2 \quad \text{sum of residues}$$

$$g = p_1 + p_2 \quad \text{sum of poles} \quad x = r_1 p_2 + r_2 p_1 \quad \text{cross product}$$
Realization – Complex Poles

Matching the terms with like coefficients gives

\[ R_o = \frac{1}{Y_o} \]

\[ p + x = \frac{R_o + R_1 + R_2}{LCR_2} \]

\[ p - x = \frac{R_1 + R_2}{LCR_2} \]

\[ 2p = \frac{R_o + 2R_1 + 2R_2}{LCR_2} \]

\[ 2x = \frac{R_o}{LCR_2} \]
Realization from S-Parameters

Solving gives

\[ L = -\frac{R_o}{2a} \]

\[ R_2 = \frac{R_o \left( \frac{p}{x} - 1 \right)}{2} - R_1 \]

\[ R_1 = \frac{1}{2} \left( \frac{gR_o}{a} + \frac{2Lx}{a} - R_o \right) \]

\[ C = \frac{-a}{R_2 x} \]
Typical SPICE Netlist

* 32-pole approximation
* This subcircuit has 16 pairs of complex poles and 0 real poles

.subckt sample 8000 9000
vsens8001 8000 8001 0.0
vsens9001 9000 9001 0.0
.subcircuit for s[1][1]
* complex residue-pole pairs for k=1 residue: -6.4662e-002 8.1147e-002 pole: -4.4593e-001 -2.4048e+001
e1c1 1 0 8001 0 1.0
hc2 2 1 vsens8001 50.0
r1erc3 2 3 50.0
vp4 3 4 0.0
l1cd5 4 5 1.933e-007
r0cd5 4 0 5.000e+001
r1cd6 5 6 5.895e+003
c1cd6 6 0 3.474e-015
r2cd6 6 0 -9.682e+003
:
* constant term 2.2 -6.192e-003
edee397 397 0 9001 0 1.0e+000
hdeee398 398 397 vsens9001 50.0
rteree399 399 399 50.0
vp400 399 400 0.0
rdee400 400 0 49.4
* current sources
fs4 0 8001 vp4 -1.0
gs4 0 8001 4 0 0.020
fs10 0 8001 vp10 -1.0
gs10 0 8001 10 0 0.020
fs16 0 8001 vp16 -1.0
gs16 0 8001 16 0 0.020
fs22 0 8001 vp22 -1.0
gs22 0 8001 22 0 0.020
fs28 0 8001 vp28 -1.0
gs28 0 8001 28 0 0.020
Realization from Y-Parameters

Recursive convolution

SPICE realization
Realization from S-Parameters

Recursive convolution

SPICE realization