ECE 546
Lecture -13
Scattering Parameters

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Transfer Function Representation

Use a two-terminal representation of system for input and output
Y-parameter Representation

\[ I_1 = y_{11}V_1 + y_{12}V_2 \]

\[ I_2 = y_{21}V_1 + y_{22}V_2 \]
Y Parameter Calculations

To make $V_2 = 0$, place a short at port 2

\[
y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0} \quad y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2=0}
\]
Z Parameters

\[ V_1 = z_{11}I_1 + z_{12}I_2 \]

\[ V_2 = z_{21}I_1 + z_{22}I_2 \]
Z-parameter Calculations

\[ Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0} \quad Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0} \]

To make \( I_2 = 0 \), place an open at port 2
$V_1 = h_{11} I_1 + h_{12} V_2$

$I_2 = h_{21} I_1 + h_{22} V_2$
H Parameter Calculations

To make $V_2 = 0$, place a short at port 2

$$h_{11} = \frac{V_1}{I_1} \bigg|_{V_2=0} \quad h_{21} = \frac{I_2}{I_1} \bigg|_{V_2=0}$$
G Parameters

\[ I_1 = g_{11}V_1 + g_{12}I_2 \]
\[ V_2 = g_{21}V_1 + g_{22}I_2 \]
G-Parameter Calculations

\[ g_{11} = \frac{I_1}{V_1} \bigg|_{I_2=0} \quad g_{21} = \frac{V_2}{V_1} \bigg|_{I_2=0} \]

To make \( I_2 = 0 \), place an open at port 2
TWO-PORT NETWORK REPRESENTATION

Z Parameters
\[ V_1 = Z_{11}I_1 + Z_{12}I_2 \]
\[ V_2 = Z_{21}I_1 + Z_{22}I_2 \]

Y Parameters
\[ I_1 = Y_{11}V_1 + Y_{12}V_2 \]
\[ I_2 = Y_{21}V_1 + Y_{22}V_2 \]

- At microwave frequencies, it is more difficult to measure total voltages and currents.

- Short and open circuits are difficult to achieve at high frequencies.

- Most active devices are not short- or open-circuit stable.
Wave Approach

Use a travelling wave approach

\[ V_1 = E_{i1} + E_{r1} \quad V_2 = E_{i2} + E_{r2} \]

\[ I_1 = \frac{E_{i1} - E_{r1}}{Z_o} \quad I_2 = \frac{E_{i2} - E_{r2}}{Z_o} \]

- Total voltage and current are made up of sums of forward and backward traveling waves.

- Traveling waves can be determined from standing-wave ratio.
Wave Approach

\[ a_1 = \frac{E_{i1}}{\sqrt{Z_o}} \quad a_2 = \frac{E_{i2}}{\sqrt{Z_o}} \]

\[ b_1 = \frac{E_{r1}}{\sqrt{Z_o}} \quad b_2 = \frac{E_{r2}}{\sqrt{Z_o}} \]

\( Z_o \) is the reference impedance of the system

\[ b_1 = S_{11} a_1 + S_{12} a_2 \]
\[ b_2 = S_{21} a_1 + S_{22} a_2 \]
Wave Approach

\[
S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2=0} \\
S_{12} = \left. \frac{b_1}{a_2} \right|_{a_1=0} \\
S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2=0} \\
S_{22} = \left. \frac{b_2}{a_2} \right|_{a_1=0}
\]

To make \( a_i = 0 \)
1) Provide no excitation at port i
2) Match port i to the characteristic impedance of the reference lines.

**CAUTION**: \( a_i \) and \( b_i \) are the traveling waves in the reference lines.
S-Parameters of TL

\[ S_{11} = S_{22} = \frac{(1 - X^2) \Gamma}{1 - X^2 \Gamma^2} \]

\[ S_{12} = S_{21} = \frac{(1 - \Gamma^2) X}{1 - X^2 \Gamma^2} \]

\[ \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \]

\[ \Gamma = \frac{Z_c - Z_{\text{ref}}}{Z_c + Z_{\text{ref}}} \]

\[ X = e^{-\gamma l} \]

\[ Z_c = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \]
S-Parameters of Lossless TL

\[ \beta = \omega \sqrt{LC} \]

\[ Z_c = \sqrt{\frac{L}{C}} \]

\[ S_{11} = S_{22} = \frac{(1 - X^2) \Gamma}{1 - X^2 \Gamma^2} \]

\[ S_{12} = S_{21} = \frac{(1 - \Gamma^2) X}{1 - X^2 \Gamma^2} \]

\[ \Gamma = \frac{Z_c - Z_{\text{ref}}}{Z_c + Z_{\text{ref}}} \]

\[ X = e^{-j\beta l} \]

If \( Z_c = Z_{\text{ref}} \)

\[ S_{11} = S_{22} = 0 \]

\[ S_{12} = S_{21} = e^{-j\beta l} \]
N-Port S Parameters

\[
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n
\end{bmatrix} = \begin{bmatrix}
  S_{11} & S_{12} & \cdots \\
  S_{21} & S_{22} & \cdots \\
  \vdots & \vdots & \ddots \\
  \vdots & \vdots & \ddots & S_{nn}
\end{bmatrix} \begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_n
\end{bmatrix}
\]

\[b = Sa\]

If \(b_i = 0\), then no reflected wave on port \(i \Rightarrow\) port is matched

\[a_i = \frac{V_i^+}{\sqrt{Z_{oi}}} \quad V_i^+ : \text{incident voltage wave in port } i\]

\[b_i = \frac{V_i^-}{\sqrt{Z_{oi}}} \quad V_i^- : \text{reflected voltage wave in port } i \quad Z_{oi} : \text{impedance in port } i\]
N-Port S Parameters

\[ v = \sqrt{Z_o} (a + b) \quad (1) \]
\[ i = \frac{1}{\sqrt{Z_o}} (a - b) \quad (2) \]
\[ v = Z i \quad (3) \]

Substitute (1) and (2) into (3)

\[ \sqrt{Z_o} (a + b) = Z \frac{1}{\sqrt{Z_o}} (a - b) \]

Defining S such that \( b = S a \) and substituting for b

\[ Z_o (U + S) a = Z_o (U - S) a \]

\[ U : \text{unit matrix} \]

\[ S \rightarrow Z \]
\[ Z = Z_o (U + S)(U - S)^{-1} \]

\[ Z \rightarrow S \]
\[ S = (Z + Z_o U)^{-1} (Z - Z_o U) \]
N-Port S Parameters

If the port reference impedances are different, we define $k$ as

$$k = \begin{bmatrix} \sqrt{Z_{o1}} \\ \sqrt{Z_{o2}} \\ \sqrt{Z_{on}} \end{bmatrix}.$$

$$v = k(a + b) \quad \text{and} \quad i = k^{-1}(a - b) \quad \text{and} \quad k(a + b) = Zk^{-1}(a - b)$$

$Z \rightarrow S$

$$S = \left( Zk^{-1} + k \right) \left( Zk^{-1} - k \right)$$

$S \rightarrow Z$

$$Z = k(U + S)(U - S)^{-1}k$$
Normalization

Assume original S parameters as $S_1$ with system $k_1$. Then the representation $S_2$ on system $k_2$ is given by

$$S_2 = \left[ k_1(U + S_1)(U - S_1)^{-1} k_{12} + k_2 \right]^{-1} \left[ k_1(U + S_1)(U - S_1)^{-1} k_{12} - k_2 \right]$$

Transformation Equation

If $Z$ is symmetric, $S$ is also symmetric
Dissipated Power

\[ P_d = \frac{1}{2} a^T (U - S^T S^*) a^* \]

The dissipation matrix \( D \) is given by:

\[ D = U - S^T S^* \]

Passivity insures that the system will always be stable provided that it is connected to another passive network.

For passivity
- (1) the determinant of \( D \) must be \( > 0 \)
- (2) the determinant of the principal minors must be \( \geq 0 \)
Dissipated Power

When the dissipation matrix is 0, we have a lossless network ➔

\[ S^T S^* = U \]

The S matrix is unitary.

For a lossless two-port:

\[ |S_{11}|^2 + |S_{21}|^2 = 1 \]
\[ |S_{22}|^2 + |S_{12}|^2 = 1 \]

If in addition the network is reciprocal, then

\[ S_{12} = S_{21} \quad \text{and} \quad |S_{11}| = |S_{22}| = \sqrt{1 - |S_{12}|^2} \]
Lossy and Dispersive Line

\[ S_{11} = S_{22} = \frac{(1 - \alpha^2) \rho}{1 - \rho^2 \alpha^2} \]

\[ S_{21} = S_{12} = \frac{(1 - \rho^2) \alpha}{1 - \rho^2 \alpha^2} \]

\[ \alpha = e^{-\gamma l} \]

\[ \rho = \frac{Z_c(\omega) - Z_o}{Z_c(\omega) + Z_o} \]
Frequency-Domain Formulation*

**Frequency-Domain**

\[
B_1(\omega) = S_{11}(\omega) A_1(\omega) + S_{12}(\omega) A_2(\omega)
\]

\[
B_2(\omega) = S_{21}(\omega) A_1(\omega) + S_{22}(\omega) A_2(\omega)
\]
Time-Domain Formulation
Time-Domain Formulation

\[ b_1(t) = s_{11}(t)^*a_1(t) + s_{12}(t)^*a_2(t) \]

\[ b_2(t) = s_{21}(t)^*a_1(t) + s_{22}(t)^*a_2(t) \]

\[ a_1(t) = \Gamma_1(t)b_1(t) + T_1(t)g_1(t) \]

\[ a_2(t) = \Gamma_2(t)b_2(t) + T_2(t)g_2(t) \]

\[ T_i(t) = \frac{Z_o}{Z_i(t) + Z_o} \]

\[ \Gamma_i(t) = \frac{Z_i(t) - Z_o}{Z_i(t) + Z_o} \]
Time-Domain Solutions

\[ a_1(t) = \frac{\left[ 1 - \Gamma_2(t)s'_{22}(0) \right]\left[ T_1(t)g_1(t) + \Gamma_1(t)M_1(t) \right]}{\Delta(t)} \]

\[ + \frac{\left[ \Gamma_1(t)s'_{12}(0) \right]\left[ T_2(t)g_2(t) + \Gamma_2(t)M_2(t) \right]}{\Delta(t)} \]

\[ a_2(t) = \frac{\left[ 1 - \Gamma_1(t)s'_{11}(0) \right]\left[ T_2(t)g_2(t) + \Gamma_2(t)M_2(t) \right]}{\Delta(t)} \]

\[ + \frac{\left[ \Gamma_2(t)s'_{21}(0) \right]\left[ T_1(t)g_1(t) + \Gamma_1(t)M_1(t) \right]}{\Delta(t)} \]
Time-Domain Solutions

\[ b_1(t) = s'_{11}(0)a_1(t) + s'_{12}(0)a_2(t) + M_1(t) \]

\[ b_2(t) = s'_{21}(0)a_1(t) + s'_{22}(0)a_2(t) + M_2(t) \]

\[ \Delta(t) = \left[ 1 - \Gamma_1(t)s'_{11}(0) \right] \left[ 1 - \Gamma_2(t)s'_{22}(0) \right] - \Gamma_1(t)s'_{12}(0)\Gamma_2(t)s'_{21}(0) \]

\[ M_1(t) = H_{11}(t) + H_{12}(t) \]

\[ M_2(t) = H_{21}(t) + H_{22}(t) \]

\[ s'_{ij}(0) = s_{ij}(0)\Delta \tau \]

\[ H_{ij}(t) = \sum_{\tau=1}^{t-1} s_{ij}(t-\tau)a_j(\tau)\Delta \tau \]
Special Case – Lossless Line

\[
s_{11}(t) = s_{22}(t) = 0 \quad s_{12}(t) = s_{21}(t) = \delta\left(t - \frac{L}{v}\right)
\]

\[
M_1(t) = a_2 \left(t - \frac{L}{v}\right) \quad M_2(t) = a_1 \left(t - \frac{L}{v}\right)
\]

\[
a_1(t) = T_1(t)g_1(t) + \Gamma_1(t)a_2 \left(t - \frac{L}{v}\right)
\]

\[
a_2(t) = T_2(t)g_2(t) + \Gamma_2(t)a_1 \left(t - \frac{L}{v}\right)
\]

\[
b_1(t) = a_2 \left(t - \frac{L}{v}\right) \quad b_2(t) = a_1 \left(t - \frac{L}{v}\right)
\]

Wave Shifting Solution
Time-Domain Solutions

\[ v_1(t) = a_1(t) + b_1(t) \]

\[ v_2(t) = a_2(t) + b_2(t) \]

\[ i_1(t) = \frac{a_1(t)}{Z_o} - \frac{b_1(t)}{Z_o} \]

\[ i_2(t) = \frac{a_2(t)}{Z_o} - \frac{b_2(t)}{Z_o} \]
Simulations

Line length = 1.27m

$Z_o = 73 \, \Omega$

$v = 0.142 \, \text{m/ns}$
Simulations

Near End

Volts

Time (ns)

Far End

Volts

Time (ns)
Simulations

Line length = 25 in  \[ L = 539 \text{ nH/m} \]

C = 39 pF/m

Pulse magnitude = 4V

\[ R_o = 1 \text{ k\Omega (GHz)}^{1/2} \]

Pulse width = 20 ns

Rise and fall times = 1ns
N-Line S-Parameters*

\[
B_1 = S_{11} A_1 + S_{12} A_2 \\
B_2 = S_{21} A_1 + S_{22} A_2
\]

Scattering Parameters for N-Line

\[ S_{21} = S_{12} = 2E_0E^{-1} \left[ 1 - \Gamma \right] \Psi \left[ 1 - \Gamma \Psi \Gamma \Psi \right]^{-1} T \]

\[ S_{11} = S_{22} = T^{-1} \left[ \Gamma - \Psi \Gamma \Psi \right] \left[ 1 - \Gamma \Psi \Gamma \Psi \right]^{-1} T \]

\[ \Gamma = \left[ 1 + EE_0^{-1} Z_0 H_0 H^{-1} Z_m^{-1} \right]^{-1} \left[ 1 - EE_0^{-1} Z_0 H_0 H^{-1} Z_m^{-1} \right] \]

\[ T = \left[ 1 + EE_0^{-1} Z_0 H_0 H^{-1} Z_m^{-1} \right]^{-1} EE_0^{-1} \]

\[ \Psi = W(-l) \]
Scattering Parameter Matrices

\( \mathbf{E}_0 \) : Reference system voltage eigenvector matrix

\( \mathbf{E} \) : Test system voltage eigenvector matrix

\( \mathbf{H}_0 \) : Reference system current eigenvector matrix

\( \mathbf{H} \) : Test system current eigenvector matrix

\( \mathbf{Z}_0 \) : Reference system modal impedance matrix

\( \mathbf{Z}_m \) : Test system modal impedance matrix
Eigen Analysis

* Diagonalize ZY and YZ and find eigenvalues.
* Eigenvalues are complex: $\lambda_i = \alpha_i + j\beta_i$

$$W(u) = \begin{bmatrix} e^{\alpha_1 u + j\beta_1 u} \\ \cdot \\ e^{\alpha_n u + j\beta_n u} \end{bmatrix}$$
Solution

\[ V_m = EV \]

\[ I_m = HI \]

\[ V_m(x) = \left[ W(-x)A + W(x)B \right] \]

\[ I_m(x) = Z_m^{-1} \left[ W(-x)A + W(x)B \right] \]

\[ Z_m = \Lambda_m^{-1}EZH^{-1} \]

\[ Z_c = E^{-1}Z_mH = E^{-1}\Lambda_m^{-1}EZ \]
Solutions

\[ a_1(t) = \Delta_1^{-1} \left[ 1 - \Gamma_1(t)s'_{11}(0) \right]^{-1} \left[ T_1(t)g_1(t) + \Gamma_1(t)M_1(t) \right] \]
\[ - \Delta_1^{-1} \left[ 1 - \Gamma_1(t)s'_{11}(0) \right]^{-1} \left[ 1 - \Gamma_2(t)s'_{22}(0) \right]^{-1} \times \]
\[ \left[ \Gamma_1(t)s'_{21}(0) \right] \left[ T_2(t)g_2(t) + \Gamma_2(t)M_2(t) \right] \]

\[ a_2(t) = \Delta_2^{-1} \left[ 1 - \Gamma_2(t)s'_{22}(0) \right]^{-1} \left[ T_2(t)g_2(t) + \Gamma_2(t)M_2(t) \right] \]
\[ - \Delta_2^{-1} \left[ 1 - \Gamma_2(t)s'_{22}(0) \right]^{-1} \left[ 1 - \Gamma_1(t)s'_{11}(0) \right]^{-1} \times \]
\[ \left[ \Gamma_1(t)s'_{12}(0) \right] \left[ T_1(t)g_1(t) + \Gamma_1(t)M_1(t) \right] \]
Solutions

\[ \Delta_1(t) = 1 - \left[ 1 - \Gamma_1(t)s'_{11}(0) \right]^{-1} \left[ 1 - \Gamma_2(t)s'_{22}(0) \right]^{-1} \Gamma_1(t)s'_{21}(0) \Gamma_2(t)s'_{12}(0) \]

\[ \Delta_2(t) = 1 - \left[ 1 - \Gamma_2(t)s'_{22}(0) \right]^{-1} \left[ 1 - \Gamma_1(t)s'_{11}(0) \right]^{-1} \Gamma_2(t)s'_{12}(0) \Gamma_1(t)s'_{21}(0) \]

\[ b_1(t) = s'_{11}(0)a_1(t) + s'_{12}(0)a_2(t) + M_1(t) \]

\[ b_2(t) = s'_{21}(0)a_1(t) + s'_{22}(0)a_2(t) + M_2(t) \]
Solutions

\[ v_{m1}(t) = a_1(t) + b_1(t) \Rightarrow v_1(t) = E_o^{-1}[a_1(t) + b_1(t)] \]

\[ v_{m2}(t) = a_2(t) + b_2(t) \Rightarrow v_2(t) = E_o^{-1}[a_2(t) + b_2(t)] \]
Lossless Case – Wave Shifting

\[ s_{21}(t) = s_{12}(t) = \delta(t - \tau_m) \]

\[ M_1(t) = a_2(t - \tau_m) \]

\[ M_2(t) = a_1(t - \tau_m) \]

\[ a_1(t) = T_1(t)g_1(t) + \Gamma_1(t)a_2(t - \tau_m) \]

\[ a_2(t) = T_2(t)g_3(t) + \Gamma_3(t)a_1(t - \tau_m) \]

\[ b_1(t) = a_2(t - \tau_m) \]

\[ b_2(t) = a_2(t - \tau_m) \]
Solution for Lossless Lines

\[
\delta(t - \tau_m) = \begin{cases} 
\delta(t - \tau_{m1}) & \\
\delta(t - \tau_{m2}) & \\
\vdots & \\
\delta(t - \tau_{mn}) & 
\end{cases}
\]

\[
a_i(t - \tau_{m}) = \begin{bmatrix} 
a_1(t - \tau_{m1}) \\
a_2(t - \tau_{m2}) \\
\vdots \\
a_n(t - \tau_{mn}) 
\end{bmatrix}
\]
Why Use S Parameters?

Y-Parameter

\[ Y_{11} = \frac{1 + e^{-2\gamma l}}{Z_c (1 - e^{-2\gamma l})} \]

- \( Z_c \): microstrip characteristic impedance
- \( \gamma \): complex propagation constant
- \( l \): length of microstrip

\( Y_{11} \) can be unstable

S-Parameter

\[ S_{11} = \frac{(1 - e^{-2\gamma l})\Gamma}{1 - \Gamma^2 e^{-2\gamma l}} \]

\[ \Gamma = \frac{Z_c - Z_o}{Z_c + Z_o} \]

\( S_{11} \) is always stable
Choice of Reference

\[ \Gamma = \frac{Z_c - Z_{\text{ref}}}{Z_c + Z_{\text{ref}}} \]

\[ Z_c = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \]

\[ Z_{\text{ref}} \text{ is arbitrary} \]

What is the best choice for \( Z_{\text{ref}} \)?

At high frequencies

\[ Z_c \to \sqrt{\frac{L}{C}} \]

Thus, if we choose

\[ Z_{\text{ref}} = \sqrt{\frac{L}{C}} \]

\[ S_{12} \to e^{-j\omega \sqrt{LC}d} = X_o \]

\[ S_{11} \to 0 \]
Choice of Reference

S-Parameter measurements (or simulations) are made using a 50-ohm system. For a 4-port, the reference impedance is given by:

\[
Z_0 = \begin{bmatrix}
50.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 50.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 50.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 50.0 \\
\end{bmatrix}
\]

- \( Z \): Impedance matrix (of blackbox)
- \( S \): S-parameter matrix
- \( Z_0 \): Reference impedance
- \( I \): Unit matrix

\[
S = \left[ ZZ_0^{-1} + I \right]^{-1} \left[ ZZ_0^{-1} - I \right]
\]

\[
Z = \left[ I + S \right] \left[ I - S \right]^{-1} Z_0
\]
## Reference Transformation

**Method:** Change reference impedance from uncoupled to coupled system to get new S-parameter representation

\[
Z_o = \begin{bmatrix}
50.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 50.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 50.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 50.0 \\
\end{bmatrix}
\quad \text{Uncoupled system}
\]

\[
Z_o = \begin{bmatrix}
328.0 & 69.6 & 328.9 & 69.6 \\
69.6 & 328.8 & 69.6 & 328.9 \\
328.9 & 69.6 & 328.8 & 69.6 \\
69.6 & 328.9 & 69.6 & 328.8 \\
\end{bmatrix}
\quad \text{Coupled system}
\]

as an example...
Choice of Reference

\[ Z_0 = \]

\begin{align*}
&50.0 & 0.0 & 0.0 & 0.0 \\
&0.0 & 50.0 & 0.0 & 0.0 \\
&0.0 & 0.0 & 50.0 & 0.0 \\
&0.0 & 0.0 & 0.0 & 50.0 \\
\end{align*}

as reference…

\[ Z_0 = \]

\begin{align*}
&328.0 & 69.6 & 328.9 & 69.6 \\
&69.6 & 328.8 & 69.6 & 328.9 \\
&328.9 & 69.6 & 328.8 & 69.6 \\
&69.6 & 328.9 & 69.6 & 328.8 \\
\end{align*}

as reference…

Harder to approximate

Easier to approximate (up to 6 GHz)
**Choice of Reference**

Using:

$$Z_0 = \begin{bmatrix} 50.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 50.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 50.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 50.0 \end{bmatrix}$$

as reference...

Using:

$$Z_0 = \begin{bmatrix} 328.0 & 69.6 & 328.9 & 69.6 \\ 69.6 & 328.8 & 69.6 & 328.9 \\ 328.9 & 69.6 & 328.8 & 69.6 \\ 69.6 & 328.9 & 69.6 & 328.8 \end{bmatrix}$$

as reference...

Harder to approximate

Easier to approximate (up to 6 GHz)
Choice of Reference

\[ Z_0 = \]

using

\[
\begin{array}{cccc}
50.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 50.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 50.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 50.0 \\
\end{array}
\]
as reference...

Easier to approximate (up to 6 GHz)

\[ Z_0 = \]

using

\[
\begin{array}{cccc}
328.0 & 69.6 & 328.9 & 69.6 \\
69.6 & 328.8 & 69.6 & 328.9 \\
328.9 & 69.6 & 328.8 & 69.6 \\
69.6 & 328.9 & 69.6 & 328.8 \\
\end{array}
\]
as reference...

Harder to approximate
Choice of Reference

\[ Z_0 = \begin{bmatrix}
328.0 & 69.6 & 328.9 & 69.6 \\
69.6 & 328.8 & 69.6 & 328.9 \\
328.9 & 69.6 & 328.8 & 69.6 \\
69.6 & 328.9 & 69.6 & 328.8
\end{bmatrix} \]

as reference...

\[ Z_0 = \begin{bmatrix}
50.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 50.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 50.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 50.0
\end{bmatrix} \]

as reference...

Easier to approximate

Harder to approximate
Choice of Reference

S11 Magnitude

- Red: \( Z_{\text{ref}} = Z_0 \)
- Blue: \( Z_{\text{ref}} = 80 \) ohms
- Black: \( Z_{\text{ref}} = 100 \) ohms

Frequency (GHz)

\( S11 \)

\( Z_{\text{ref}} = Z_0 \)
Choice of Reference

**S21 Magnitude**

- **Zref=Zo**
- **Zref=80 ohms**
- **Zref=100 ohms**

**Frequency (GHz)**

- 0.7
- 0.75
- 0.8
- 0.85
Modeling of Discontinuities

1. Tapered Lines

2. Capacitive Discontinuities
Tapered Microstrip

General topology of tapered microstrip with $d_w$: width at wide end, $d_n$: width at narrow end, $l_w$: length of wide section, $l_n$: length of narrow section, $l_t$: length of tapered section.
Tapered Line Analysis Using S Parameters*

\[ u_j(t) = s_{21}^{(j)}(t) * u_{j-1}(t) + s_{22}^{(j)}(t) * w_j(t) \]

\[ w_j(t) = s_{11}^{(j+1)}(t) * u_j(t) + s_{12}^{(j+1)}(t) * w_{j+1}(t) \]

Tapered Transmission Line

**Small End**
- Excitation at small end

**Wide End**
- Excitation at small end

**Small End**
- Excitation at wide end

**Wide End**
- Excitation at wide end
Tapered Transmission Line

Varying tapering rate

Near End

Far End
Capacitive Load

\[ Z_0 \]

\[ Z_0 \quad C \]

Near End -- \( C = 4 \text{ pF} \)

Far end -- \( C = 4 \text{ pF} \)
Capacitive Load
Multidrop Buses

- Stubs of TL with nonlinear loads
- Reduce speed and bandwidth
- Limit driving capabilities
Transmission Lines with Capacitive Discontinuities
Capacitive Discontinuity

\[ V_i + V_r = V_t \]

\[ \frac{V_i - V_r}{Z_o} + \frac{E}{R} - \frac{V_i - V_r}{R} = \frac{V_t}{Z_o} \]

\[ V_r = T_c E + \Gamma_c V_i \]
Scattering Parameter Analysis

\[ u_j(t) = s_{21}^{(j)}(t) * u_{j-1}'(t) + s_{22}^{(j)}(t) * w_j(t) \]

\[ u_j'(t) = u_j(t) + u_j''(t) \]

\[ w_j(t) = w_j'(t) + u_j''(t) \]
Capacitive Loading
Computer-simulated near end responses for capacitively loaded transmission line with $l = 3.6$ in, $w = 8$ mils, $h = 5$ mils. Pulse parameters are $V_{\text{max}} = 4$ V, $tr = tf = 0.5$ ns, $tw = 4$ ns. Left: Varying $P$ with $C = 2$ pF. Right: Varying $C$ with $P = 300$ mils.